

## Applications of Linear Approximation

In this unit we're trying to learn about applications of the derivative to real problems. Here is one such example that involves math as well as physics.

### Example 4: Planet Quirk

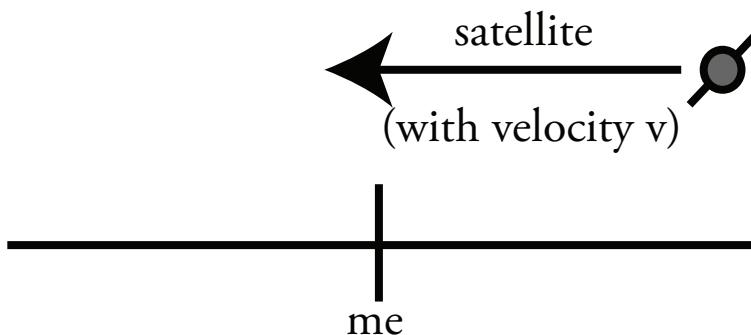


Figure 1: Illustration of Example 4: a satellite with velocity  $v$  speeding past “me” on planet Quirk.

Let's say I am on Planet Quirk, and that a satellite is whizzing overhead with a velocity  $v$ . The satellite has a clock on it that reports a time  $T$ . I have a watch that reports a time  $T_m$ . We want to calculate the time dilation (a concept from special relativity) that describes the difference between  $T$  and  $T_m$ .

We borrow the following equation from special relativity:

$$T_m = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Again,  $T_m$  is the time I measure on my wristwatch, and  $T$  is the time measured on board the satellite. How different are  $T_m$  and  $T$ ?

To avoid dividing by a square root, we'll once again use linear approximation:

$$\frac{T}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{T}{\sqrt{1 - u}} \approx T \left( 1 + \frac{1}{2}u \right),$$

where  $u = \frac{v^2}{c^2}$ . (If you're wondering why we got  $(1 + \frac{1}{2}u)$  and not  $(1 - \frac{1}{2}u)$ , try applying the change of variables  $v = -u$  before approximating.)

And so, we find that:

$$T_m = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}} \approx T \left( 1 + \frac{1}{2} \left( \frac{v^2}{c^2} \right) \right)$$

How does this affect you in the real world? GPS transmitters are mounted on satellites, the satellites are moving, and you might be moving too. According

to special relativity, there will be a difference between the time on your GPS device and the time on the satellite. This time difference will affect the GPS device's estimate of your position.

The engineers who set up the GPS satellite system knew this and had to decide if they needed to take this into account when designing the system. It turns out that GPS satellites move at about  $v = 4$  kilometers per second (km/s) and  $c = 3 \times 10^5$  km/sec, and so  $\frac{v^2}{c^2} \approx 10^{-10}$  and  $T_m \approx T(1.0000000005)$ . There's hardly any difference between the times measured on the ground and in the satellite.

Because  $\frac{v^2}{c^2}$  is very close to zero, our linear approximation should be quite close to the actual value of  $T_m$ . Another good reason for using linear approximation here is that if the answer is "the difference is too small to matter", the person doing the calculation has no use for a more precise answer which may be more difficult to calculate.

Nevertheless, engineers used this very approximation (along with several other such approximations) to calibrate the radio transmitters on GPS satellites. The satellites transmit at a slightly offset frequency.

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