Hi. Welcome back to recitation.

In lecture you talked about computing derivatives by definition. And one rule for computing derivatives that Professor Jerison mentioned but didn't prove was what's called the constant multiple rule. So today I want to give you a proof of that rule and show you a little bit of geometric intuition for why it works.

So the constant multiple rule says that if you have a constant c in a differentiable function, f of x, that the derivative of the function c times f of x is equal to c times the derivative of f of x.

Just to do a quick example, suppose that c were 3 and f of x were the function x squared, this says that the derivative d by dx of 3 x squared is equal to 3 times the derivative of d by dx of x squared. Now, this is good because we already have a rule for computing derivatives of powers of x. So this says we don't need a special rule for computing multiples of powers of x squared. We don't need to go back to the limit definition to compute the derivative of 3 x squared. We can just use the fact that we know the derivative of x squared in order to compute the derivative of 3 x squared. So in this case that would work out to 6x. In this case.

So it simplifies the number of different computations you have to do. It reduces the number of times we need to go back to the limit definition. So that's the use of the rule. Let's quickly talk about its proof.

The idea behind the proof is you have these two derivatives and you want to show that they're equal. Well, any time you have a derivative, what it really means is it's the value of some limit of some difference quotient. So in this case we have the derivative d by dx of c times f of x by definition is the limit of a difference quotient as delta x goes to 0 of-- so we take the function c times f of x and we plug in x plus delta x and we plug in x and we take the difference and we divide by delta x. So that's c times f of x plus delta x minus c times f of x divided by delta x.

Now you'll notice that here both terms in the numerator have this constant factor, c, in them. So we can factor that out. And I'll just pull it out in front of this whole fraction so that this is the limit as delta x goes to 0 of c times the ratio f of x plus delta x minus f of x, all quantity over delta x.

Now, c is just some constant. This part depends on delta x. And on x, but on delta x. So as delta x goes to zero, this changes while this stays the same. What that means is, so as delta x goes to 0, this gets closer and closer to something, the value of its limit. And c, you're just multiplying it in, so c times-- the limit of c times this is equal to c times whatever the limit of this is. If this is getting closer and closer to some value, c times it is getting closer and closer to c times that value.

So this is equal to c-- in other words, we can pull constant multiples outside of limits. So this limit as delta x-- c times the limit is delta x goes to 0 of f of x plus delta x minus f of x over delta x. And this limit here is just the definition of the derivative of f of x. So this is equal to, by definition, c times d by dx of f of x.

So we started with the derivative of c times f of x and we showed this is equal to c times the derivative of f of x. That's exactly what we wanted. So that proves the constant multiple rule.

We've now proved the constant multiple rule-- let me talk a little bit about some geometric intuition for why this works.

So I've got here, well so, you know, let's take c equals 2, just for simplicity. So here I have a graph y equals f of x, and I have also drawn the graph, y equals 2f of x. The relationship between these graphs is that y equals 2f of x is what you get when you stretch the graph for y equals f of x vertically by a factor of 2. So, you know, if it passed through 0 before, it still passes through 0. But everywhere else, if it was above 0, it's now twice as high. If it was below 0, it's now twice as low.

So if you think about what the definition, what the derivative means in terms of this graph geometrically, it's telling you the limit-- sorry, the slope of a tangent line. Or in other words, the limit of the slopes of secant lines.

So if you look at these two curves, say-- let's pick a couple values of x, say, and then maybe x plus delta x-- so if you look at the secant lines for these two curves through those points, what you see is that these two lines, they have the same-- you know, so the slope of a line is its rise over its run-- so they have the same run, that we are talking about the same little interval, here. But this, in the function that's scaled up, in the y equals 2f of x curve, we have that that the rise-- everything has been stretched upwards by a factor of two-- so the rise here is exactly double the rise here. So the slope of the secant line is exactly double the slope of this secant line.

And similarly, the tangent line-- just a limit of secant lines-- has been stretched by that same factor of two. So the slope of the tangent line is exactly twice the slope of the tangent line for the other function. So the tangent line here is exactly twice as steep as the tangent line here. Or in other words, the derivative of this function is exactly twice the derivative of that function.

So that's just a geometric statement of this very same constant multiple rule that we stated algebraically at the beginning and that we just proved. So that's that.