18.014 Problem Set 8 Solutions

Total: 24 points

Problem 1: Compute

$$\int_0^1 xf''(2x)dx$$

given that f'' is continuous for all x, and $f(0) = 1$, $f'(0) = 3$, $f(1) = 5$, $f'(1) = 2$, $f(2) = 7$, $f'(2) = 4$.

$\boxed{\text{Solution}}$ (4 points) Applying integration by parts (theorem 5.5), we have

$$\int_0^1 xf''(2x)dx = \frac{1}{2}xf'(2x)\Big|_0^1 - \frac{1}{2}\int_0^1 f'(2x)dx = \frac{1}{2}f'(2) - \frac{1}{4}f(2) + \frac{1}{4}f(0) = \frac{1}{2}.$$

We can use this theorem because x is differentiable with constant derivative 1 that is continuous and never changes sign, and $f''(2x)$ is continuous by hypothesis.

Problem 2: Use the definition $a^x = e^{x \log a}$ to derive the following properties of general exponentials:
(b) $(ab)^x = a^x b^x$.
(c) $a^x a^y = a^{x+y}$.
(d) $(a^x)^y = (a^y)^x = a^{xy}$
(e) Suppose $a > 0$, $a \neq 1$. Then $y = a^x$ if and only if $x = \log_a y$.

$\boxed{\text{Solution}}$ (4 points) (b) By the definition of the exponential function, part (ii) of theorem 3 of course notes M, part (i) of theorem 2 of course notes M, and the definition of the exponential function, we have

$$(ab)^x = e^{x\log(ab)} = e^{x\log(a)+x\log(b)} = e^{x\log(a)}e^{x\log(b)} = a^x b^x.$$

(c) By the definition of the exponential function, part (i) of theorem 2 of course notes M, and the definition of the exponential function, we have

$$a^x a^y = e^{x\log(a)}e^{y\log(a)} = e^{(x+y)\log(a)} = a^{x+y}.$$

(d) After twice using the definition of the exponential function, using that the exponential function and the logarithmic function are inverses, and again using the definition of the exponential function, we obtain

$$(a^x)^y = e^{y\log(a^x)} = e^{y\log(e^{x\log a})} = e^{yx\log(a)} = a^{xy}.$$

The same argument with the roles of x and y interchanged yields

$$(a^y)^x = e^{x\log(a^y)} = e^{x\log(e^{y\log a})} = e^{xy\log(a)} = a^{xy}.$$

Combining the two yields the statement in part (d).
(e) Suppose $y = a^x$. By the definition of the exponential function, we know $y = e^{x\log(a)}$. Taking the logarithm of both sides and using that the logarithm and the exponential function are inverses, we obtain $\log(y) = x\log(a)$. Since $a \neq 1$, $\log(a) \neq 0$. Thus, we may divide by $\log(a)$ to get $x = \frac{\log(y)}{\log(a)}$. But, this is our definition of $\log_a y$. Writing this argument backwards implies the converse statement.

Problem 3: (a) Use integration by parts to deduce the formula

$$\int \sin^2(x)dx = -\sin(x)\cos(x) + \int \cos^2(x)dx.$$

In the second integral, write $\cos^2(x) = 1 - \sin^2(x)$ and thereby deduce the formula

$$\int \sin^2(x)dx = \frac{1}{2}x - \frac{1}{4}\sin(2x).$$

(b) Use integration by parts and the result of part (a) to deduce

$$\int x\sin^2(x)dx = \frac{1}{4}x^2 - \frac{1}{4}x\sin(2x) - \frac{1}{8}\cos(2x).$$

$\boxed{\text{Solution}}$ (4 points) (a) Putting $f(x) = g'(x) = \sin(x)$ in formula (5.23) and using $\frac{d}{dx}\sin(x) = \cos(x)$, $\int \sin(x) = -\cos(x)$ yields

$$\int \sin^2(x)dx = -\sin(x)\cos(x) + \int \cos^2(x)dx.$$

Now, if we substitute $\cos^2(x) = 1 - \sin^2(x)$ and use part (e) of theorem 3 of course notes L, then our expression becomes

$$\int \sin^2(x)dx = -\frac{1}{2}\sin(2x) + \int 1dx - \int \sin^2(x)dx.$$

Integrating $\int 1dx$, moving the $\int \sin^2(x)dx$ to the other side of the equation, and dividing by 2 yields

$$\int \sin^2(x)dx = -\frac{1}{4}\sin(2x) + \frac{1}{2}x.$$

(b) Putting $f(x) = x$ and $g'(x) = \sin^2(x)$ in formula (5.23), we get

$$\int x \sin^2(x) = x \int \sin^2(x) - \int \int \sin^2(x).$$

Using part (a), we obtain

$$\begin{aligned} \int x \sin^2(x) &= x \left(-\frac{1}{4} \sin(2x) + \frac{1}{2} x \right) - \int \left(-\frac{1}{4} \sin(2x) + \frac{1}{2} x \right) \\ &= -\frac{1}{4} x \sin(2x) + \frac{1}{4} x^2 - \frac{1}{8} \cos(2x). \end{aligned}$$

Problem 4: Evaluate the integral

$$\int \sqrt{1 + 3\cos^2(x)} \sin(2x) dx.$$

$\boxed{\text{Solution}}$ (4 points) Put $u = 1 + 3\cos^2(x)$. Then $du = -6\cos(x)\sin(x)dx = -3\sin(2x)dx$ by part (e) of section L of the course notes. Applying the method of substitution, we obtain

$$\frac{-1}{3} \int \sqrt{u} du = \frac{-2}{9} u^{3/2}.$$

Plugging back in for u, our answer is

$$\frac{-2}{9} \left(1 + 3\cos^2(x)\right)^{3/2}.$$

Problem 5: (a) Find a polynomial $P(x)$ such that $P'(x) - 3P(x) = 4 - 5x + 3x^2$. Prove there is only one solution.
(b) If $Q(x)$ is a given polynomial, prove that there is one and only one polynomial $P(x)$ such that $P'(x) - 3P(x) = Q(x)$.

$\boxed{\text{Solution}}$ (4 points) (a) Put $P(x) = -x^2 + x - 1$. Then $P'(x) = -2x + 1$ and $P'(x) - 3P(x) = 3x^2 - 5x + 4$. Note that $\deg(P'(x) - 3P(x)) = \deg P(x)$; hence $\deg P(x) = 3$. If $P(x) = ax^2 + bx + c$, then $P'(x) = 2ax + b$ and

$$P'(x) - 3P(x) = -3ax^2 + (2a - 3b)x + (b - 3c) = 3x^2 - 5x + 4.$$

Because there is an unique solution to the equations $-3a = 3$, $2a - 3b = -5$, and $b - 3c = 4$, our solution must be the only one.

(b) First, we show there is at most one solution to the equation $P'(x) - 3P(x) = Q(x)$. If P_1 and P_2 are two distinct solutions, then $P_1'(x) - 3P_1(x) = P_2'(x) - 3P_2(x)$. In particular, $(P_1 - P_2)' = -3(P_1 - P_2)$. However, if $P_1 - P_2$ is not constant, then $\deg(P_1 - P_2) + 1 = \deg((P_1 - P_2)')$. Since two polynomials must have the same degree if they are equal, we deduce that $P_1 - P_2$ is constant. Clearly, this constant must be zero and $P_1 = P_2$, a contradiction. We conclude that there can be at most one solution to our equation.
Now, we show that there exists a solution to the equation $P'(x) - 3P(x) = Q(x)$. We proceed by induction on $\deg Q$. For the base case $\deg Q = 0$, we may take $P = -\frac{1}{3}Q$. Now for the inductive step. Suppose the statement is true for all polynomials Q of degree $k < n$. We will prove the statement for all polynomials Q of degree n.
Let $Q = c_n x^n + Q_1$ be a polynomial of degree n where Q_1 is a polynomial of degree at most $n-1$. Now, by the induction hypothesis, we may find a solution P_1 to the equation $P_1' - 3P_1 = Q_1 - \frac{1}{3}nc_n x^{n-1}$ since $Q_1 - \frac{1}{3}nc_n x^{n-1}$ is a polynomial of degree at most $n-1$. Let $P = P_1 - \frac{1}{3}c_n x^n$. Then

$$P' - 3P = P_1' - 3P_1' - \frac{1}{3}nc_n x^{n-1} + c_n x^n = Q_1 + c_n x^n = Q.$$

The desired result follows.

Problem 6: Evaluate

$$\int \frac{x^4+2}{x^4+x^3+x^2}dx.$$

$\boxed{\text{Solution}}$ (4 points) Note $x^4 + x^3 + x^2 = x^2(x^2 + x + 1)$. Using partial fractions, we observe

$$\frac{x^4+2}{x^4+x^3+x^2} = 1 - \frac{x^3+x^2-2}{x^2(x^2+x+1)} = 1 - \frac{2x-2}{x^2} - \frac{-x+1}{x^2+x+1}.$$

Note $\int 1 = x$, $\int \frac{2x-2}{x^2} = 2\log|x| + 2x^{-1}$. To evaluate the last term, we write

$$\int \frac{-x+1}{x^2+x+1} = -\frac{1}{2}\log|x^2+x+1| + \frac{1}{2}\int \frac{1}{x^2+x+1}.$$

Now, we write

$$\frac{1}{x^2+x+1} = \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}.$$

Using the identity on the top of page 263, we find

$$\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}} = \frac{4}{3}\arctan\left(\frac{4(x+\frac{1}{2})}{3}\right).$$

Combining all of these terms, we obtain

$$\int \frac{x^4+2}{x^4+x^3+x^2}dx = x - 2\log(x) - 2x^{-1} + \frac{1}{2}\log|x^2+x+1| - \frac{2}{3}\arctan\left(\frac{4(x+\frac{1}{2})}{3}\right).$$

Bonus: Let f be a continuous function. Prove that

$$\int_0^x \frac{f(u)(x-u)^n}{n!}du = \int_0^x\left(\int_0^x\left(\cdots\left(\int_0^{u_2}\left(\int_0^{u_1} f(t)dt\right)du_1\right)\cdots\right)du_n\right).$$

$\boxed{\text{Solution}}$ (4 points) We prove the statement by induction on n. The base case $n=0$ is the tautology

$$\int_0^x f(u)du = \int_0^x f(t)dt.$$

Now, assume that the statement is true for $n-1$. We prove the statement for n. Integration by parts yields

$$\int_0^x \frac{f(u)(x-u)^n}{n!} = \left(\int_0^u f(t)dt\right)\frac{(x-u)^n}{n!}\Big|_0^x + \int_0^x\left(\int_0^u f(t)dt\right)\frac{(x-u)^{n-1}}{(n-1)!}.$$

The first term is zero. Define $g(u) = \int_0^u f(t)dt$. Then we may apply the inductive hypothesis to obtain

$$\int_0^x g(u)\frac{(x-u)^{n-1}}{(n-1)!}du = \int_0^x\left(\int_0^{u_n}\left(\cdots\left(\int_0^{u_2} g(u_1)du_1\right)\cdots\right)du_n\right).$$

Plugging in for g, we get

$$\int_0^x \frac{f(u)(x-u)^n}{n!} = \int_0^x\left(\int_0^{u_n}\left(\cdots\left(\int_0^{u_2}\left(\int_0^{u_1} f(t)dt\right)du_1\right)\cdots\right)du_n\right)$$

as desired.