PROFESSOR: To begin today I want to remind you, I need to write it down on the board at least twice, of the fundamental theorem of calculus. We called it FTC 1 because it's the first version of the fundamental theorem. We'll be talking about another version, called the second version, today. And what it says is this: If $F' = f$, then the integral from a to b of $f(x)\, dx$ is equal to $F(b) - F(a)$. So that's the fundamental theorem of calculus. And the way we used it last time was, this was used to evaluate integrals. Not surprisingly, that's how we used it. But today, I want to reverse that point of view. We're going to read the equation backwards, and we're going to write it this way. And we're going to use f to understand capital F. Or in other words, the derivative to understand the function.

So that's the reversal of point of view that I'd like to make. And we'll make this point in various ways. So information about f, about F', gives us information about F. Now, since there were questions about the mean value theorem, I'm going to illustrate this first by making a comparison between the fundamental theorem of calculus and the mean value theorem. So we're going to compare this fundamental theorem of calculus with what we call the mean value theorem. And in order to do that, I'm going to introduce a couple of notations. I'll write delta F as $F(b) - F(a)$. And another highly imaginative notation, delta $x = b - a$. So here's the change in F, there's the change in x. And then, this fundamental theorem can be written, of course, right up above there is the formula. And it's the formula for delta F. So this is what we call the fundamental theorem of calculus.

I'm going to divide by delta x, now. And If I divide by delta x, that's the same thing as $1 / (b-a)$ times the integral from a to b of $f(x)\, dx$. So I've just rewritten the formula here. And this expression here, on the right-hand side, is a fairly important one. This is the average of f. That's the average value of f. Now, so this is going to permit me to make the comparison between the mean value theorem, which we don't have stated yet here. And the fundamental theorem. And I'll do it in the form of inequalities. So right in the middle here, I'm going to put the fundamental theorem. It says that delta F in this notation is equal to, well if I multiply by delta x again, I can write it as the average of f-- So I'm going to write it as the average of F'

here. Times delta x. So we have this factor here, which is the average of F', or the average of little f, it's the same thing. And then I multiplied through again. So I put the thing in the red box, here.

STUDENT: [INAUDIBLE]

PROFESSOR: Isn't what the average of big F?

So the question is, why is this the average of little f rather than the average of big F. So the average of a function is the typical value. If, for example, little f were constant, little f were constant, then this integral would be-- So the question is why is this the average. And I'll take a little second to explain that. But I think I'll explain it over here. Because I'm going to erase it. So the idea of an average is the following. For example, imagine that a = 0 and b = n, let's say for example. And so we might sum the function from 1 to n. Now, that would be the sum of the values from 1 to n. But the average is, we divide by n here. So this is the average. And this is a kind of Riemann sum, representing the integral from 0 to n, of f(x) dx. Where the increment, delta x, is 1.

So this is the notion of an average value here, but in the continuum setting, as opposed to the discrete setting. Whereas what's on the left-hand side is the change in F. The capital F. And this is the average of the little f. So an average is a sum. And it's like an integral. So, in other words what I have here is that the change in F is the average of its infinitesimal change times the amount of time elapsed, if you like. So this is the statement of the fundamental theorem. Just rewritten. Exactly what I wrote there. But I multiplied back by delta x. Now, let me compare this with the mean value theorem. The mean value theorem also is an equation. The mean value theorem says that this is equal to F'(c) delta x. Now, I pulled a fast one on you. I used capital F's here to make the analogy clear. But the role of the letter is important to make the transition to this comparison. We're talking about the function capital F here. And its derivative.

Now, this is true. So now I claim that this thing is fairly specific. Whereas this, unfortunately, is a little bit vague. And the reason why it's vague is that c is just somewhere in the interval. So some c-- Sorry, this is some c, in between a and b. So really, since we don't know where this thing is, we don't know which of the values it is, we can't say what it is. All we can do is say, well for sure it's less than the largest value, say, the maximum of F', times delta x. And the only thing we can say for sure on the other end is that it's less than or equal to-- sorry, it's

greater than or equal to the minimum of F' times delta x. Over that same interval. This is over 0 less than-- sorry, $a < x < b$. So that means that the fundamental theorem of calculus is a much more specific thing. And indeed it gives the same conclusion. It's much stronger than the mean value theorem. It's way better than the mean value theorem. In fact, as soon as we have integrals, we can abandon the mean value theorem. We don't want it. It's too simple-minded. And what we have is something much more sophisticated, which we can use. Which is this. So it's obvious that if this is the average, the average is less than the maximum. So it's obvious that it works just as well to draw this conclusion. And similarly over here with the minimum. OK, the average is always bigger than the minimum and smaller than the max.

So this is the connection, if you like. And I'm going to elaborate just one step further by talking about the problem that you had on the exam. So there was an Exam 2 problem. And I'll show you how it works using the mean value theorem and how it works using integrals. But I'm going to have to use this notation capital F. So capital F', as opposed to the little f, which was what was the notation that was on your exam. So we had this situation here. These were the givens of the problem. And then the question was, the mean value theorem says, or implies, if you like, it doesn't say it, but it implies it - implies A is less than capital F of 4 is less than B, for which A and B?

So let's take a look at what it says. Well, the mean value theorem says that F(F(4) - F(0) = F'(c) (4 - 0). This is this F' times delta x, this is the change in x. And that's the same thing as 1/(1+c) times 4. And so the range of values of this number here is from / 1/(1+0) times 4, that's 4. To, that's the largest value, to the smallest that it gets, which is 1/(1+4) times 4. That's the range. And so the conclusion is that F(4) - f(0) is between, well, let's see. It's between 4 and 4/5. Which are those two numbers down there. And if you remember that F(0) was 1, this is the same F(4) is between 5 and 9/5. So that's the way that you were supposed to solve the problem on the exam.

On the other hand, let's compare to what you would do with the fundamental theorem of calculus. With the fundamentals theorem of calculus, we have the following formula. F(4) - F(0) is equal to the integral from 0 to 4 of dx / (1+x). That's what the fundamental theorem says. And now I claim that we can get these same types of results by a very elementary observation. It's really the same observation that I made up here, that the average is less than or equal to the maximum. Which is that the biggest this can ever be is, let's see. The biggest it is when x is 0, that's 1. So the biggest it ever gets is this. And that's equal to 4. Right? On the

other hand, the smallest it ever gets to be, it's equal to this. The smallest it ever gets to be is the integral from 0 to 4 of 1/5 dx. Because that's the lowest value that the integrand takes. When x = 4, it's 1/5. And that's equal to 4/5. Now, there's a little tiny detail which is that really we know that this is the area of some rectangle and this is strictly smaller. And we know that these inequalities are actually strict. But that's a minor point. And certainly not one that we'll pay close attention to.

But now, let me show you what this looks like geometrically. So geometrically, we interpret this as the area under a curve. Here's a piece of the curve y = 1/(1+x). And it's going up to 4 and starting at 0 here. And the first estimate that we made - that is, the upper bound - was by trapping this in this big rectangle here. We compared it to the constant function, which was 1 all the way across. This is y = 1. And then we also trapped it from underneath by the function which was at the bottom. And this was y = 1/5. And so what this really is is, these things are the simplest possible Riemann sum. Sort of a silly Riemann sum. This is a Riemann sum with one rectangle. This is the simplest possible one. And so this is a very, very crude estimate. You can see it misses by a mile. The larger and the smaller values are off by a factor of 5. But this one is called the-- this one is the lower Riemann sum. And that one is less than our actual integral. Which is less than the upper Riemann sum. And you should, by now, have looked at those upper and lower sums on your homework. So it's just the rectangles underneath and the rectangles on top.

So at this point, we can literally abandon the mean value theorem. Because we have a much better way of getting at things. If we chop things up into more rectangles, we'll get much better numerical approximations. And if we use simpleminded expressions with integrals, we'll be able to figure out any bound we could get using the mean value theorem. So that's not the relevance of the mean value theorem. I'll explain to you why we talked about it, even, in a few minutes. OK, are there any questions before we go on? Yeah.

STUDENT: [INAUDIBLE]

PROFESSOR: I knew that the range of c was from 0 to 4, I should have said that right here. This is true for this theorem. The mean value theorem comes with an extra statement, which I missed. Which is that this is for some c between 0 and 4. So I know the range is between 0 and 4. The reason why it's between 0 and 4 is that's part of the mean value theorem. We started at 0, we ended at 4. So the c has to be somewhere in between. That's part of the mean value theorem.

STUDENT: [INAUDIBLE]

PROFESSOR: The question is, do you exclude any values that are above 4 and below 0. Yes, absolutely. The point is that in order to figure out how F changes, capital F changes, between 0 and 4, you need only pay attention to the values in between. You don't have to pay any attention to what the function is doing below 0 or above 4. Those things are strictly irrelevant.

STUDENT: [INAUDIBLE]

PROFESSOR: Yeah, I mean it's strictly in between these two numbers. I have to understand what the lowest and the highest one is.

STUDENT: [INAUDIBLE]

PROFESSOR: It's approaching that, so.

OK. So now, the next thing that we're going to talk about is, since I've got that 1 up there, that Fundamental Theorem of Calculus 1, I need to talk about version 2. So here is the Fundamental Theorem of Calculus version 2. I'm going to start out with a function little f, and I'm going to assume that it's continuous. And then I'm going to define a new function, which is defined as a definite integral. G(x) is the integral from a to x of f(t) dt. Now, I want to emphasize here because it's the first time that I'm writing something like this, that this is a fairly complicated gadget. It plays a very basic and very fundamental but simple role, but it nevertheless is a little complicated. What's happening here is that the upper limit I've now called x, and the variable t is ranging between a and x, and that the a and the x are fixed when I calculate the integral. And the t is what's called the dummy variable. It's the variable of integration. You'll see a lot of people who will mix this x with this t. And if you do that, you will get confused, potentially hopelessly confused, in this class. In 18.02 you will be completely lost if you do that. So don't do it. Don't mix these two guys up. It's actually done by many people in textbooks, and it's fairly careless. Especially in old-fashioned textbooks. But don't do it.

So here we have this G(x). Now, remember, this G(x) really does make sense. If you give me an a, and you give me an x, I can figure out what this is, because I can figure out the Riemann sum. So of course I need to know what the function is, too. But anyway, we have a numerical procedure for figuring out what the function G is. Now, as is suggested by this mysterious letter x being in the place where it is, I'm actually going to vary x. So the conclusion is that if this is true, and this is just a parenthesis, not part of the theorem. It's just an indication of what

the notation means. Then G' = f.

Let me first explain what the significance of this theorem is, from the point of view of differential equations. G(x) solves the differential equation y' = f(x). So y' = f, I shouldn't put the x in if I got it here, with the condition y(a) = 0. So it solves this pair of conditions here. The rate of change, and the initial position is specified here. Because when you integrate from a to a, you get 0 always. And what this theorem says is you can always solve that equation. When we did differential equations, I said that already. I said we'll treat these as always solved. Well, here's the reason. We have a numerical procedure for computing things like this. We could always solve this equation. And the formula is a fairly complicated gadget, but so far just associated with Riemann sums.

Alright, now. Let's just do one example. Unfortunately, not a complicated example and maybe not persuasive as to why you would care about this just yet. But nevertheless very important. Because this is the quiz question which everybody gets wrong until they practice it. So the integral from, say 1 to x, of dt / t^2. Let's try this one here. So here's an example of this theorem, I claim. Now, this is a question which challenges your ability to understand what the question means. Because it's got a lot of symbols. It's got the integration and it's got the differentiation. However, what it really is is an exercise in recopying. You look at it and you write down the answer. And the reason is that, by definition, this function in here is a function of the form G(x) of the theorem over here. So this is the G(x). And by definition, we said that G'(x) = f(x). Well, what's the f(x)? Look inside here. It's what's called the integrand. This is the integral from 0 to x of f(t) dt, right? Where the f(t) is equal to 1 / t^2. So your ability is challenged. You have to take that 1 / t^2 and you have to plug in the letter x, instead of t, for it. And then write it down. As I say, this is an exercise in recopying what's there.

So this is quite easy to do, right? I mean, you just look and you write it down. But nevertheless, it looks like a long, elaborate object here. Pardon me?

STUDENT: [INAUDIBLE]

PROFESSOR: So the question was, why did I integrate.

STUDENT: [INAUDIBLE]

PROFESSOR: Why did I not integrate? Ah. Very good question. Why did I not integrate. The reason why I didn't integrate is I didn't need to. Just as when you take the antiderivative-- sorry, the

derivative of something, you take the antiderivative, you get back to the thing. So, in this case, we're taking the antiderivative of something and we're differentiating. So we end back in the same place where we started. We started with f(t), we're ending with f. Little f. So you integrate, and then differentiate. And you get back to the same place. Now, the only difference between this and the other version is, in this case when you differentiate and integrate you could be off by a constant. That's what that shift, why there are two pieces to this one. But there's never an extra piece here. There's no plus c here. When you integrate and differentiate, you kill whatever the constant is. Because the derivative of a constant is 0. So no matter what the constant is, hiding inside of G, you're getting the same result. So this is the basic idea. Now, I just want to double-check it, using the Fundamental Theorem of Calculus 1 here.

So let's actually evaluate the integral. So now I'm going to do what you've suggested, which is I'm just going to check whether it's true. No, no I am because I'm going just double-check that it's consistent. It certainly is slower this way, and we're not going to want to do this all the time, but we might as well check one. So this is our integral. And we know how to do it. No, I need to do it. And this is $-t^{-1}$, evaluated at 1 and x. Again, there's something subliminally here for you to think about. Which is that, remember, it's t is ranging between 1 and t = x. And this is one of the big reasons why this letter t has to be different from x. Because here it's 1 and there it's x. It's not x. So you can't put an x here. Again, this is t = 1 and this is t = x over there. And now if I plug that in, I get what? I get $-1/x$, and then I get $-(-1)$. So this is, let me get rid of those little t's there. This is a little easier to read. And so now let's check it. It's d/dx. So here's what G(x) is. $G(x) = 1 - 1/x$. That's what G(x) is. And if I differentiate that, I get $+1 / x^2$. That's it. You see the constant washed away.

So now, here's my job. My job is to prove these theorems. I never did prove them for you. So, I'm going to prove the Fundamental Theorem of Calculus. But I'm going to do 2 first. And then I'm going to do 1. And it's just going to take me just one blackboard. It's not that hard. The proof is by picture. And, using the interpretation as area under the curve. So if here's the value of a, and this is the graph of the function y equals f of x. Then I want to draw three vertical lines. One of them is going to be at x. And one of them is going to be at x + delta x. So here I have the interval from 0 to x, and next I have the interval from x to delta x more than that. And now the pieces that I've got are the area of this part. So this has area which has a name. It's called G(x). By definition, G(x), which is sitting right over here in the fundamental theorem, is the integral from a to x of this function. So it's the area under the curve. So that area is G(x).

Now this other chunk here, I claim that this is delta G. This is the change in G. It's the value of G(x) that is the area of the whole business all the way up to x + delta x minus the first part, G(x). So it's what's left over. It's the incremental amount of area there.

And now I am going to carry out a pretty standard estimation here. This is practically a rectangle. And it's got a base of delta x, and so we need to figure out what its height is. This is delta G, and it's approximately its base times its height. But what is the height? Well, the height is maybe either this segment or this segment or something in between. But they're all about the same. So I'm just going to put in the value at the first point. That's the left end there. So that's this height here, is f(x). So this is f(x), and so really I approximate it by that rectangle there. And now if I divide and take the limit, as delta x goes to 0, of delta G / delta x, it's going to equal f(x). And this is where I'm using the fact that f is continuous. Because I need the values nearby to be similar to the value in the limit. OK, that's the end. This the end of the proof, so I'll put a nice little Q.E.D. here.

So we've done Fundamental Theorem of Calculus 2, and now we're ready for Fundamental Theorem of Calculus 1. So now I still have it on the blackboard to remind you. It says that the integral of the derivative is the function, at least the difference between the values of the function at two places. So the place where we start is with this property that F' = f. That's the starting-- that's the hypothesis. Now, unfortunately, I'm going to have to assume something extra in order to use the Fundamental Theorem of Calculus 2, which is I'm going to assume that f is continuous. That's not really necessary, but that's just a very minor technical point, which I'm just going to ignore. So we're going to start with F' = f. And then I'm going to go somewhere else. I'm going to define a new function, G(x), which is the integral from a to x of f(t) dt. This is where we needed all of the labor of Riemann sums. Because otherwise we don't have a way of making sense out of what this even means.

So hiding behind this one sentence is the fact that we actually have a number. We have a formula for such functions. So there is a function G(x) which, once you've produced a little f for me, I can cook up a function capital G for you. Now, we're going to apply this Fundamental Theorem of Calculus 2, the one that we've already checked. So what does it say? It says that G' = f. And so now we're in the following situation. We know that F'(x) = G'(x). That's what we've got so far. And now we have one last step to get a good connection between F and G. Which is that we can conclude that F(x) is G(x) plus a constant.

Now, this little step may seem innocuous but I remind you that this is the spot that requires the

mean value theorem. So in order not to lie to you, we actually tell you what the underpinnings of all of calculus are. And they're this: the fact, if you like, that if two functions have the same derivative, they differ by a constant. Or that if a function has derivative 0, it's a constant itself. Now, that is the fundamental step that's needed, the underlying step that's needed. And, unfortunately, there aren't any proofs of it that are less complicated than using the mean value theorem. And so that's why we talk a little bit about the mean value theorem, because we don't want to lie to you about what's really going on. Yes.

STUDENT: [INAUDIBLE]

PROFESSOR: The question is how did I get from here, to here. And the answer is that if G' is little f, and we also know that F' is little f, then F' is G'. OK. Other questions?

Alright, so we're almost done. I just have to work out the arithmetic here. So I start with F(b) - F(a). And that's equal to (G(b) + c) - (G(a) + c). And then I cancel the c's. So I have here G(b) - G(a). And now I just have to check what each of these is. So remember the definition of G here. G(b) is just what we want. The integral from a to b of f(x) dx. Well I called it f(t) dt, that's the same as f(x) dx now, because I have the limit being b and I'm allowed to use x as the dummy variable. Now the other one, I claim, is 0. Because it's the integral from a to a. This one is the integral from a to a. Which gives us 0. So this is just this minus 0, and that's the end. That's it. I started with F(b) - F(a), I got to the integral. Question?

STUDENT: [INAUDIBLE]

PROFESSOR: How did I get from F(b) - F(a), is (G(b) + c) - (G(a) + c), that's the question.

STUDENT: [INAUDIBLE]

PROFESSOR: Oh, sorry this is an equals sign. Sorry, the second line didn't draw. OK, equals. Because we're plugging in, for F(x), the formula for it. Yes.

STUDENT: [INAUDIBLE]

PROFESSOR: This step here? Or this one?

STUDENT: [INAUDIBLE]

PROFESSOR: Right. So that was a good question. But the answer is that that's the statement that we're aiming for. That's the Fundamental Theorem of Calculus 1, which we don't know yet. So we're

trying to prove it, and that's why we haven't, we can't assume it. OK, so let me just notice that in the example that we had, before we go on to something else here. In the example above, what we had was the following thing. We had, say, $F(x) = -1/x$. So $F'(x) = 1 / x^2$. And, say, $G(x) = 1 - 1/x$. And you can see that either way you do that, if you integrate from 1 to 2, let's say, which is what we had over there, dt / t^2, you're going to get either $-1/t$, 1 to 2, or, if you like, $1 - 1/t$, 1 to 2. So this is the F version, this is the G version. And that's what plays itself out here, in this general proof.

Alright. So now I want to go back to the theme for today, which is using little f to understand capital F. In other words, using the derivative of F to understand capital F. And I want to illustrate it by some more complicated examples. So I guess I just erased it, but we just took the antiderivative of $1 / t^2$. And there's-- all of the powers work easily, but one. And the tricky one is the power $1 / x$. So let's consider the differential equation $L'(x) = 1 / x$. And say, with the initial value $L(1) = 0$. The solution, so the Fundamental Theorem of Calculus 2 tells us the solution is this function here. $L(x)$ equals the integral from 1 to x, dt / t. That's how we solve all such equations. We just integrate, take the definite integral. And I'm starting at 1 because I insisted that $L(1)$ be 0. So that's the solution to the problem. And now the thing that's interesting here is that we started from a polynomial. Or we started from a rational, a ratio of polynomials; that is, $1 / t$ or $1 / x$. And we get to a function which is actually what's known as a transcendental function. It's not an algebraic function. Yeah, question.

STUDENT: [INAUDIBLE]

PROFESSOR: The question is why is this equal to that. And the answer is, it's for the same reason that this is equal to that. It's the same reason as this. It's that the 1's cancel. We've taken the value of something at 2 minus the value at 1. The value at 2 minus the value at 1. And you'll get a 1 in the one case, and you get a 1 in the other case. And you subtract them and they will cancel. They'll give you 0. These two things really are equal. This is not a function evaluated at one place, it's the difference between the function evaluated at 2 and the value at 1. And whenever you subtract two things like that, constants drop out.

STUDENT: [INAUDIBLE] PROFESSOR: That's right. If I put 2 here, if I put c here, it would have been the same. It would just have dropped out. It's not there. And that's exactly this arithmetic right here. It doesn't matter which antiderivative you take. When you take the differences, the c's will cancel. You always get the same answer in the end. That's exactly why I wrote this down, so that you would see that. It doesn't matter which one you do.

So, we still have a couple of minutes left here. This is actually-- So let me go back. So here's the antiderivative of 1 / x, with value 1 at 0. Now, in disguise, we know what this function is. We know this function is the logarithm function. But this is actually a better way of deriving all of the formulas for the logarithm. This is a much quicker and more efficient way of doing it. We had to do it by very laborious processes. This will allow us to do it very easily. And so, I'm going to do that next time. But rather than do that now, I'm going to point out to you that we can also get truly new functions.

OK, so there are all kinds of new functions. So this is the first example of this kind would be, for example, to solve the equation $y' = e^{-x^2}$ with $y(0) = 0$, let's say. Now, the solution to that is a function which again I can write down by the fundamental theorem. It's the integral from 0 to x of e^{-t^2} dt. This is a very famous function. This shape here is known as the bell curve. And it's the thing that comes up in probability all the time. This shape e^{-x^2}. And our function is geometrically just the area under the curve here. This is $F(x)$. If this place is x. So I have a geometric definition, I have a way of constructing what it is by Riemann sums. And I have a function here. But the curious thing about $F(x)$ is that $F(x)$ cannot be expressed in terms of any function you've seen previously. So logs, exponentials, trig functions, cannot be. It's a totally new function.

Nevertheless, we'll be able to get any possible piece of information we would want to, out of this function. It's perfectly acceptable function, it will work just great for us. Just like any other function. Just like the log. And what this is analogous to is the following kind of thing. If you take the circle, the ancient Greeks, if you like, already understood that if you have a circle of radius 1, then its area is pi. So that's a geometric construction of what you could call a new number. Which is outside of the realm of what you might expect. And the weird thing about this number pi is that it is not the root of an algebraic equation with rational coefficients. It's what's called transcendental. Meaning, it's just completely outside of the realm of algebra. And, indeed, the logarithm function is called a transcendental function, because it's completely out of the realm of algebra. It's only in calculus that you come up with this kind of thing. So these kinds of functions will have access to a huge class of new functions here, all of which are important tools in science and engineering. So, see you next time.