18.01 Single Variable Calculus
Fall 2006

Lecture 37: Taylor Series

General Power Series

What *is* $\cos x$ anyway?

Recall: geometric series

$$1 + a + a^2 + \cdots = \frac{1}{1-a} \quad \text{for } \mid a \mid < 1$$

General power series is an infinite sum:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

represents f when $|x| < R$ where $R =$ radius of convergence. This means that for $|x| < R$, $|a_n x^n| \to 0$ as $n \to \infty$ ("geometrically"). On the other hand, if $|x| > R$, then $|a_n x^n|$ does not tend to 0. For example, in the case of the geometric series, if $|a| = \frac{1}{2}$, then $|a^n| = \frac{1}{2^n}$. Since the higher-order terms get increasingly small if $|a| < 1$, the "tail" of the series is negligible.

Example 1. If $a = -1, |a^n| = 1$ does not tend to 0.

$$1 - 1 + 1 - 1 + \cdots$$

The sum bounces back and forth between 0 and 1. Therefore it does not approach 0. Outside the interval $-1 < a < 1$, the series diverges.

Basic Tools

Rules of polynomials apply to series within the radius of convergence.

Substitution/Algebra

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

Example 2. x = -u.

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots$$

Example 3. $x = -v^2$.

$$\frac{1}{1+v^2} = 1 - v^2 + v^4 - v^6 + \cdots$$

Example 4.

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right) = (1+x+x^2+\cdots)(1+x+x^2+\cdots)$$

Term-by-term multiplication gives:

$$1+2x+3x^2+\cdots$$

Remember, here x is some number like $\frac{1}{2}$. As you take higher and higher powers of x, the result gets smaller and smaller.

Differentiation (term by term)

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[1+x+x^2+x^3+\cdots\right]$$

$$\frac{1}{(1-x)^2} = 0+1+2x+3x^2+\cdots \quad \text{where 1 is } a_0, 2 \text{ is } a_1 \text{ and 3 is } a_2$$

Same answer as Example 4, but using a new method.

Integration (term by term)

$$\int f(x)\,dx = c + \left(a_0 + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots\right)$$

where

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Example 5. $\int \frac{du}{1+u}$

$$\left(\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots\right)$$

$$\int \frac{du}{1+u} = c + u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots$$

$$\ln(1+x) = \int_0^x \frac{du}{1+u} = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$$

So now we know the series expansion of $\ln(1+x)$.

Example 6. Integrate Example 3.

$$\frac{1}{1+v^2} = 1 - v^2 + v^4 - v^6 + \cdots$$

$$\int \frac{dv}{1+v^2} = c + \left(v - \frac{v^3}{3} + \frac{v^5}{5} - \frac{v^7}{7} + \cdots\right)$$

$$\tan^{-1} x = \int_0^x \frac{dv}{1+v^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Taylor's Series and Taylor's Formula

If $f(x) = a_0 + a_1x + a_2x^2 + \cdots$, we want to figure out what all these coefficients are. Differentiating,

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots$$
$$f''(x) = (2)(1)a_2 + (3)(2)a_3x + (4)(3)a_4x^2 + \cdots$$
$$f'''(x) = (3)(2)(1)a_3 + (4)(3)(2)a_4x + \cdots$$

Let's plug in $x = 0$ to all of these equations.

$$f(0) = a_0;\ f'(0) = a_1;\ f''(0) = 2a_2;\ f'''(0) = (3!)a_3$$

Taylor's Formula tells us what the coefficients are:

$$\boxed{f^{(n)}(0) = (n!)a_n}$$

Remember, $n! = n(n-1)(n-2)\cdots(2)(1)$ and $0! = 1$. Coefficients a_n are given by:

$$\boxed{a_n = \left(\frac{1}{n!}\right) f^{(n)}(0)}$$

Example 7. $f(x) = e^x$.

$$f'(x) = e^x$$
$$f''(x) = e^x$$
$$f^{(n)}(x) = e^x$$
$$f^{(n)}(0) = e^0 = 1$$

Therefore, by Taylor's Formula $a_n = \dfrac{1}{n!}$ and

$$e^x = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Or in compact form,

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Now, we can calculate e to any accuracy:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Example 7. $f(x) = \cos x$.

$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$
$$f^{(4)}(x) = \cos x$$
$$f(0) = \cos(0) = 1$$
$$f'(0) = -\sin(0) = 0$$
$$f''(0) = -\cos(0) = -1$$
$$f'''(0) = \sin(0) = 0$$

Only *even* coefficients are non-zero, and their signs alternate. Therefore,

$$\boxed{\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots}$$

Note: $\cos(x)$ is an even function. So is this power series — as it contains only even powers of x.

There are two ways of finding the Taylor Series for $\sin x$. Take derivative of $\cos x$, or use Taylor's formula. We will take the derivative:

$$-\sin x = \frac{d}{dx}\cos x = 0 - 2\left(\frac{1}{2}\right)x + \frac{4}{4!}x^3 - \frac{6}{6!}x^5 + \frac{8}{8!}x^7 + \cdots$$

$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\boxed{\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

Compare with quadratic approximation from earlier in the term:

$$\boxed{\cos x \approx 1 - \frac{1}{2}x^2 \qquad \sin x \approx x}$$

We can also write:

$$\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}(-1)^k = (-1)^0\frac{x^0}{0!} + (-1)^2\frac{x^2}{2!} + \cdots = 1 - \frac{1}{2}x^2 + \cdots$$

$$\sin x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}(-1)^k \leftarrow n = 2k+1$$

Example 8: Binomial Expansion. $f(x) = (1+x)^a$

$$(1+x)^a = 1 + \frac{a}{1}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots$$

Taylor Series with Another Base Point

A Taylor series with its base point at a (instead of at 0) looks like:

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2 + \frac{f^{(3)}(b)}{3!}(x-b)^3 + ...$$

Taylor series for $\sqrt{x}$. It's a bad idea to expand using $b = 0$ because $\sqrt{x}$ is not differentiable at $x = 0$. Instead use $b = 1$.

$$x^{1/2} = 1 + \frac{1}{2}(x-1) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(x-1)^2 + \cdots$$