18.01 Single Variable Calculus
Fall 2006

Lecture 29: Partial Fractions

We continue the discussion we started last lecture about integrating rational functions. We defined a rational function as the ratio of two polynomials:

$$\frac{P(x)}{Q(x)}$$

We looked at the example

$$\int \left[\frac{1}{x-1} + \frac{3}{x+2}\right] dx = \ln|x-1| + 3\ln|x+2| + c$$

That same problem can be disguised:

$$\frac{1}{x-1} + \frac{3}{x+2} = \frac{(x+2)+3(x-1)}{(x-1)(x+2)} = \frac{4x-1}{x^2+x-2}$$

which leaves us to integrate this:

$$\int \frac{4x-1}{x^2+x-2}\, dx = ???$$

Goal: we want to figure out a systematic way to split $\frac{P(x)}{Q(x)}$ into simpler pieces.

First, we factor the denominator $Q(x)$.

$$\frac{4x-1}{x^2+x-2} = \frac{4x-1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

There's a slow way to find A and B. You can clear the denominator by multiplying through by $(x-1)(x+2)$:

$$(4x-1) = A(x+2) + B(x-1)$$

From this, you find

$$4 = A + B \quad \text{and} \quad -1 = 2A - B$$

You can then solve these simultaneous linear equations for A and B. This approach can take a very long time if you're working with 3, 4, or more variables.

There's a faster way, which we call the "cover-up method". Multiply both sides by $(x-1)$:

$$\frac{4x-1}{x+2} = A + \frac{B}{x+2}(x-1)$$

Set $x = 1$ to make the B term drop out:

$$\frac{4-1}{1+2} = A$$

$$A = 1$$

The fastest way is to do this in your head or physically *cover up* the struck-through terms. For instance, to evaluate B:

$$\frac{4x-1}{(x-1)\cancel{(x+2)}} = \cancel{\frac{A}{x-1}} + \frac{B}{\cancel{(x+2)}}$$

Implicitly, we are multiplying by $(x+2)$ and setting $x=-2$. This gives us

$$\frac{4(-2)-1}{-2-1} = B \quad \Longrightarrow \quad B = 3$$

What we've described so far works when $Q(x)$ factors completely into *distinct* factors and the degree of P is less than the degree of Q.

If the factors of Q repeat, we use a slightly different approach. For example:

$$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

Use the cover-up method on the highest degree term in $(x-1)$.

$$\frac{x^2+1}{x+2} = B + [\text{stuff}](x-1)^2 \quad \Longrightarrow \quad \frac{1^2+2}{1+2} = B \quad \Longrightarrow \quad B = 1$$

Implicitly, we multiplied by $(x-1)^2$, then took the limit as $x \to 1$.

C can also be evaluated by the cover-up method. Set $x=-2$ to get

$$\frac{x^2+2}{(x-1)}^{2} = C + [\text{stuff}](x+2) \quad \Longrightarrow \quad \frac{(-2)^2+2}{(-2-1)^2} = C \quad \Longrightarrow \quad C = \frac{2}{3}$$

This yields

$$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{1}{(x-1)^2} + \frac{2/3}{x+2}$$

Cover-up can't be used to evaluate A. Instead, plug in an easy value of x: $x=0$.

$$\frac{2}{(-1)^2(2)} = \frac{A}{-1} + 1 + \frac{1}{3} \Longrightarrow 1 = 1 + \frac{1}{3} - A \Longrightarrow A = \frac{1}{3}$$

Now we have a complete answer:

$$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{1}{3(x-1)} + \frac{1}{(x-1)^2} + \frac{2}{3(x+2)}$$

Not all polynomials factor completely (without resorting to using complex numbers). For example:

$$\frac{1}{(x^2+1)(x-1)} = \frac{A_1}{x-1} + \frac{B_1x+C_1}{x^2+1}$$

We find A_1, as usual, by the cover-up method.

$$\frac{1}{1^2+1} = A_1 \quad \Longrightarrow \quad A_1 = \frac{1}{2}$$

Now, we have

$$\frac{1}{(x^2+1)(x-1)} = \frac{1/2}{x-1} + \frac{B_1 x + C_1}{x^2+1}$$

Plug in $x = 0$.

$$\frac{1}{1(-1)} = -\frac{1}{2} + \frac{C_1}{1} \qquad \Longrightarrow \qquad C_1 = -\frac{1}{2}$$

Now, plug in any value other than $x = 0, 1$. For example, let's use $x = -1$.

$$\frac{1}{2(-2)} = \frac{1/2}{-2} + \frac{B_1(-1) - 1/2}{2} \implies 0 = -\frac{B_1 - 1/2}{2} \implies B_1 = -\frac{1}{2}$$

Alternatively, you can multiply out to clear the denominators (not done here).

Let's try to integrate this function, now.

$$\int \frac{dx}{(x^2+1)(x-1)} = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{x\,dx}{x^2+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln|\,x^2+1\,| - \frac{1}{2}\tan^{-1}x + c$$

What if we're faced with something that looks like this?

$$\int \frac{dx}{(x-1)^{10}}$$

This is actually quite simple to integrate:

$$\int \frac{dx}{(x-1)^{10}} = -\frac{1}{9}(x-1)^{-9} + c$$

What about this?

$$\int \frac{dx}{(x^2+1)^{10}}$$

Here, we would use trig substitution:

$$x = \tan u \quad \text{and} \quad dx = \sec^2 u du$$

and the trig identity

$$\tan^2 u + 1 = \sec^2 u$$

to get

$$\int \frac{\sec^2 u\,du}{(\sec^2 u)^{10}} = \int \cos^{18} u\,du$$

From here, we can evaluate this integral using the methods we introduced two lectures ago.