18.01 Single Variable Calculus
Fall 2006

Lecture 2: Limits, Continuity, and Trigonometric Limits

More about the "rate of change" interpretation of the derivative

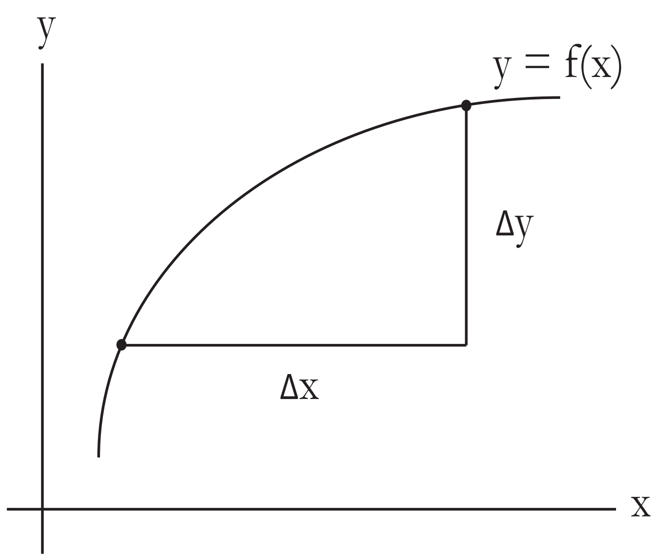

Figure 1: Graph of a generic function, with Δx and Δy marked on the graph

$$\frac{\Delta y}{\Delta x} \quad \to \quad \frac{dy}{dx} \text{ as } \Delta x \to 0$$

Average rate of change $\quad\to\quad$ Instantaneous rate of change

Examples

1. $q =$ charge $\qquad \dfrac{dq}{dt} =$ electrical current
2. $s =$ distance $\qquad \dfrac{ds}{dt} =$ speed
3. $T =$ temperature $\qquad \dfrac{dT}{dx} =$ temperature gradient

4. Sensitivity of measurements: An example is carried out on Problem Set 1. In GPS, radio signals give us h up to a certain measurement error (See Fig. 2 and Fig. 3). The question is how accurately can we measure L. To decide, we find $\frac{\Delta L}{\Delta h}$. In other words, these variables are related to each other. We want to find how a change in one variable affects the other variable.

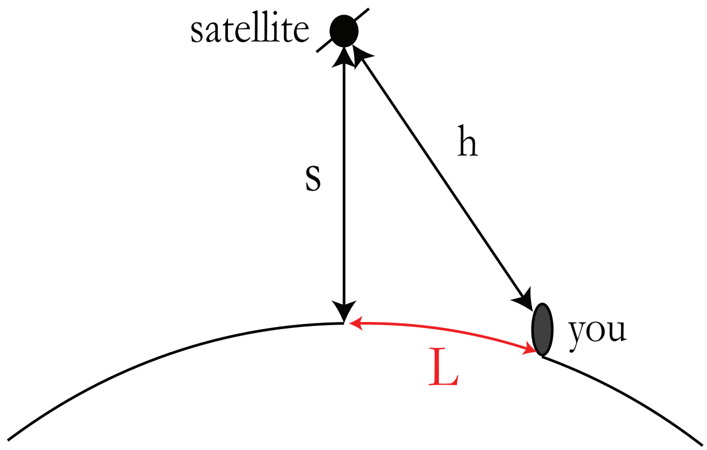

Figure 2: The Global Positioning System Problem (GPS)

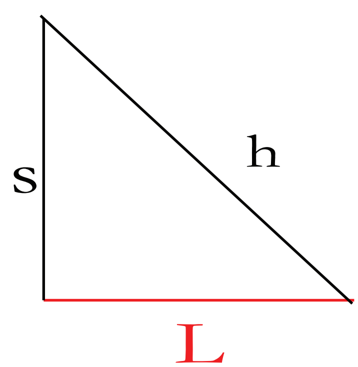

Figure 3: On problem set 1, you will look at this simplified “flat earth” model

Limits and Continuity

Easy Limits

$$\lim_{x\to 3} \frac{x^2+x}{x+1} = \frac{3^2+3}{3+1} = \frac{12}{4} = 3$$

With an easy limit, you can get a meaningful answer just by plugging in the limiting value.

Remember,

$$\lim_{x\to x_0} \frac{\Delta f}{\Delta x} = \lim_{x\to x_0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

is never an easy limit, because the denominator $\Delta x = 0$ is not allowed. (The limit $x \to x_0$ is computed under the implicit assumption that $x \neq x_0$.)

Continuity

We say $f(x)$ is continuous at x_0 when

$$\lim_{x\to x_0} f(x) = f(x_0)$$

Pictures

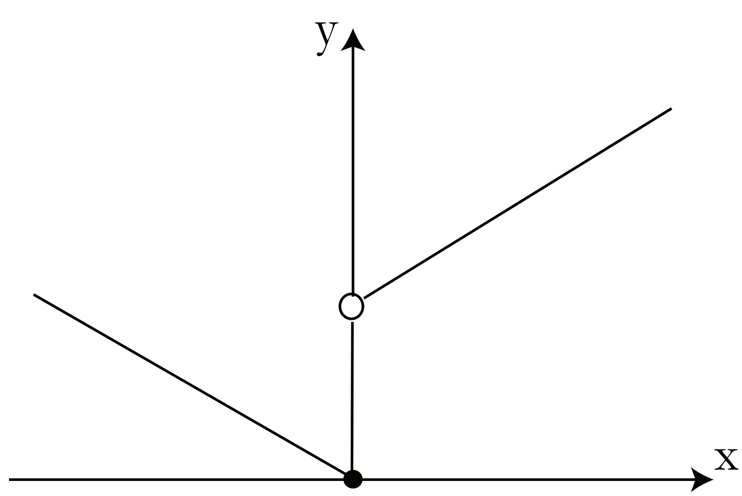

Figure 4: Graph of the discontinuous function listed below

$$f(x) = \begin{cases} x+1 & x > 0 \\ -x & x \geq 0 \end{cases}$$

This *discontinuous* function is seen in Fig. 4. For $x > 0$,

$$\lim_{x \to 0} f(x) = 1$$

but $f(0) = 0$. (One can also say, f is continuous from the left at 0, not the right.)

1. Removable Discontinuity

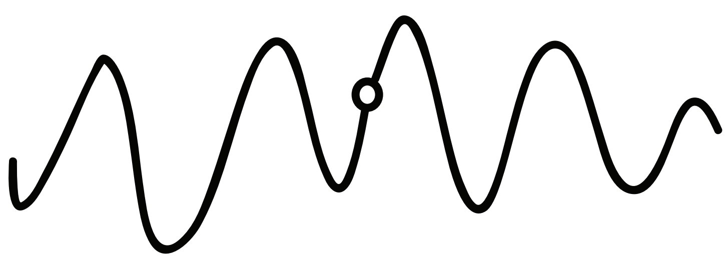

Figure 5: A removable discontinuity: function is continuous everywhere, except for one point

Definition of removable discontinuity

Right-hand limit: $\lim\limits_{x \to x_0^+} f(x)$ means $\lim\limits_{x \to x_0} f(x)$ for $x > x_0$.

Left-hand limit: $\lim\limits_{x \to x_0^-} f(x)$ means $\lim\limits_{x \to x_0} f(x)$ for $x < x_0$.

If $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0^-} f(x)$ but this is not $f(x_0)$, or if $f(x_0)$ is undefined, we say the discontinuity is *removable*.

For example, $\dfrac{\sin(x)}{x}$ is defined for $x \neq 0$. We will see later how to evaluate the limit as $x \to 0$.

2. **Jump Discontinuity**

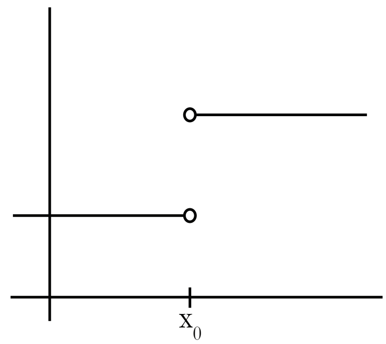

Figure 6: An example of a jump discontinuity

$\lim_{x \to x_0^+}$ for $(x < x_0)$ exists, and $\lim_{x \to x_0^-}$ for $(x > x_0)$ also exists, but they are NOT equal.

3. **Infinite Discontinuity**

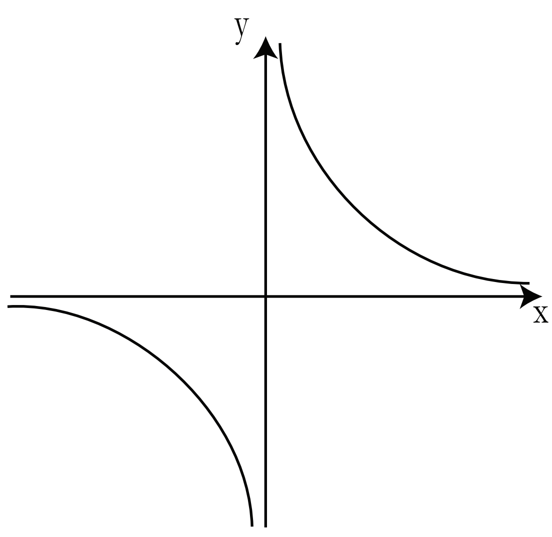

Figure 7: An example of an infinite discontinuity: $\frac{1}{x}$

Right-hand limit: $\lim_{x \to 0^+} \frac{1}{x} = \infty$; Left-hand limit: $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

4. **Other (ugly) discontinuities**

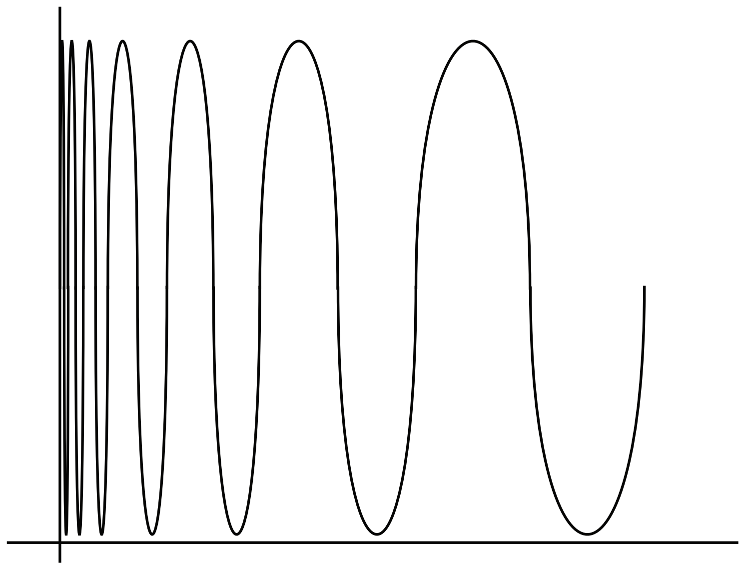

Figure 8: An example of an ugly discontinuity: a function that oscillates a lot as it approaches the origin

This function doesn't even go to $\pm\infty$ — it doesn't make sense to say it goes to anything. For something like this, we say the limit does not exist.

Picturing the derivative

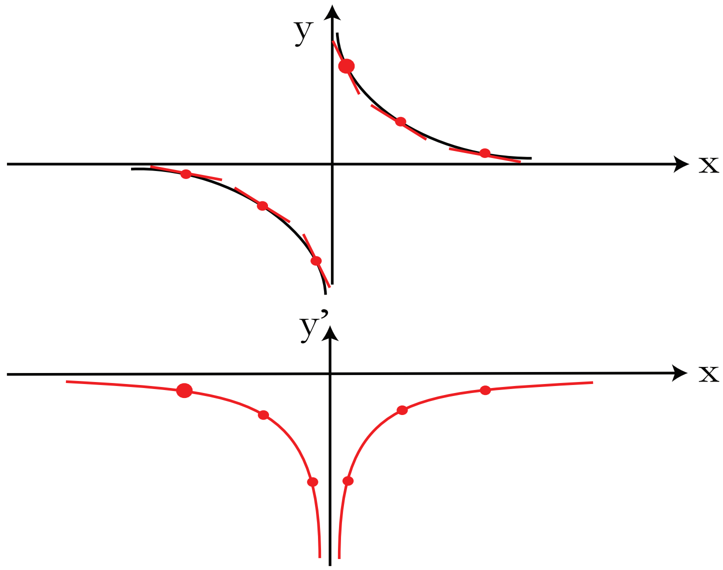

Figure 9: Top: graph of $f(x) = \frac{1}{x}$ and Bottom: graph of $f'(x) = -\frac{1}{x^2}$

Notice that the graph of $f(x)$ does NOT look like the graph of $f'(x)$! (You might also notice that $f(x)$ is an odd function, while $f'(x)$ is an even function. The derivative of an odd function is always even, and vice versa.)

Pumpkin Drop, Part II
This time, someone throws a pumpkin over the tallest building on campus.

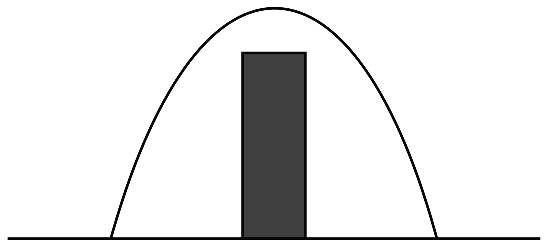

Figure 10: $y = 400 - 16t^2$, $-5 \leq t \leq 5$

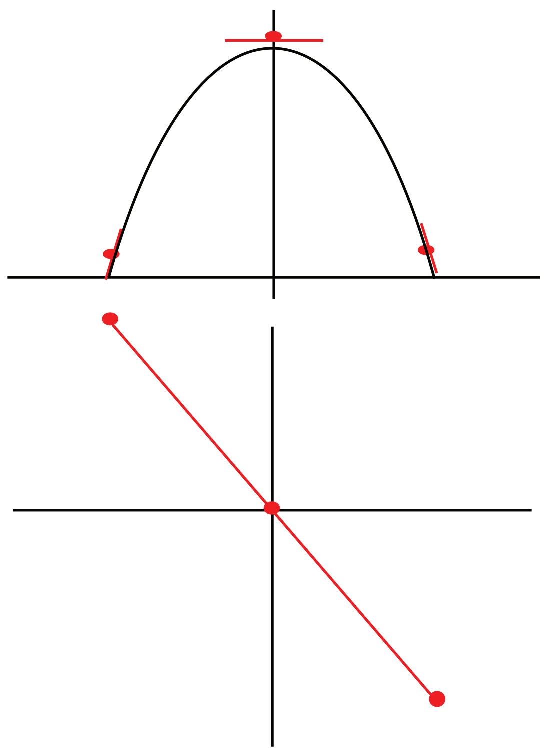

Figure 11: Top: graph of $y(t) = 400 - 16t^2$. Bottom: the derivative, $y'(t)$

Two Trig Limits

Note: In the expressions below, θ is in radians— NOT degrees!

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1; \qquad \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0}$$

Here is a geometric proof for the first limit:

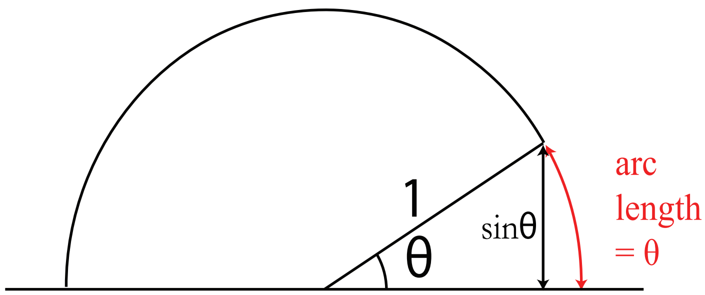

Figure 12: A circle of radius 1 with an arc of angle θ

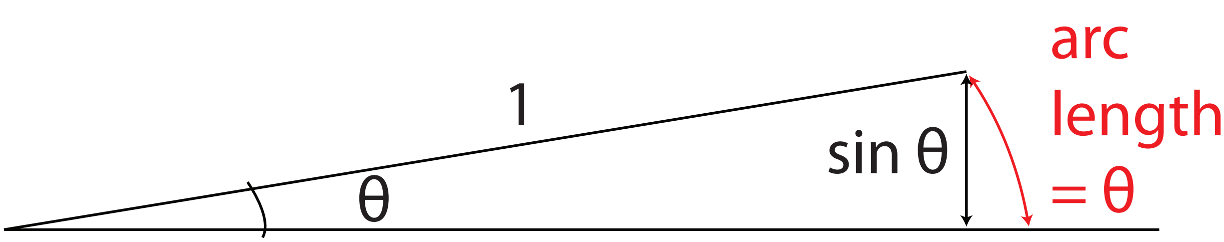

Figure 13: The sector in Fig. 12 as θ becomes very small

Imagine what happens to the picture as θ gets very small (see Fig. 13). As $\theta \to 0$, we see that $\frac{\sin\theta}{\theta} \to 1$.

What about the second limit involving cosine?

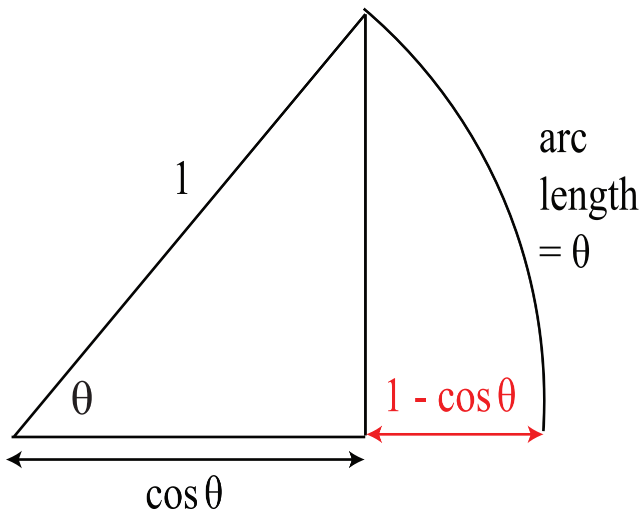

Figure 14: Same picture as Fig. 12 except that the horizontal distance between the edge of the triangle and the perimeter of the circle is marked

From Fig. 15 we can see that as $\theta \to 0$, the length $1 - \cos\theta$ of the short segment gets much smaller than the vertical distance θ along the arc. Hence, $\dfrac{1-\cos\theta}{\theta} \to 0$.

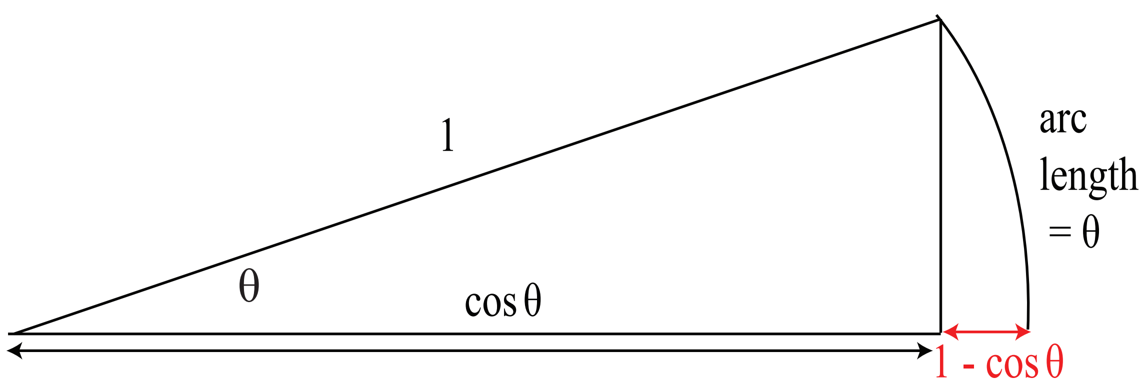

Figure 15: The sector in Fig. 14 as θ becomes very small

We end this lecture with a theorem that will help us to compute more derivatives next time.

Theorem: Differentiable Implies Continuous.
If f is differentiable at x_0, then f is continuous at x_0.

Proof: $\lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0}\right](x - x_0) = f'(x_0) \cdot 0 = 0.$

Remember: you can never divide by zero! The first step was to multiply by $\frac{x - x_0}{x - x_0}$. It looks as if this is illegal because when $x = x_0$, we are multiplying by $\frac{0}{0}$. But when computing the limit as $x \to x_0$ we always assume $x \neq x_0$. In other words $x - x_0 \neq 0$. So the proof is valid.