18.01 Single Variable Calculus
Fall 2006

18.01 Practice Final Exam

There are 19 problems, totaling 250 points. No books, notes, or calculators. This practice exam should take 3 hours.

Generally useful trigonometry:

$$\sin^2 x = \frac{1-\cos 2x}{2}; \qquad \cos^2 x = \frac{1+\cos 2x}{2}; \qquad \int \sec x = \ln(\sec x + \tan x)$$

$$\sec x = \frac{1}{\cos x}; \qquad \sin^2 x + \cos^2 x = 1; \qquad \tan^2 x + 1 = \sec^2 x$$

In a 30-60-90 right triangle, with hypotenuse 2, the legs are 1 and $\sqrt{3}$.

$$\sqrt{2} = 1.41 \qquad \sqrt{3} = 1.73 \qquad \pi = 3.14 \qquad \ln 2 = .69 \qquad \ln 10 = 2.3$$

Problem 1. (15) Evaluate each of the following:

a) $\dfrac{d}{dx}\dfrac{\ln x}{x^2}$; simplify your answer.

b) $\dfrac{d}{du}\sqrt{3\sin^2 u + 2}$

c) $\left.\dfrac{d^n}{dx^n}e^{kx}\right|_{x=0}$, k constant.

Problem 2. (10) Find the equation of the line tangent to the graph of $x^2y^2 + y^3 = 2$ at the point $(1,1)$ on the graph. (Give the equation in the form $y = mx + b$.)

Problem 3. (10) Using implicit differentiation, derive the formula for $D\cos^{-1} x$ by using the formula for $D\cos x$. (Let $y = \cos^{-1} x$.)

Problem 4. (10) Let $f(x) = \begin{cases} x^2 + x + a, & x \le 0 \\ bx + 2, & x > 0 \end{cases}$, a and b constants.
Find all values of a and b for which $f(x)$ is differentiable.

Problem 5. (15) On a night when the full moon is directly overhead, an outdoor Christmas tree 50 feet high is falling over. Its top is falling at the rate of 2 feet/sec, at the moment when it is 30 feet from the ground. At that moment, how rapidly is the shadow of the tree cast by the moon lengthening?

Problem 6. (15) Find the area of the largest rectangle whose base lies along the x-axis and whose top corners lie on the parabola $y = 1 - x^2$.

Problem 7. (15: 4,7,4) The graph of $y = y(x)$ has this property: at each point (x, y) on the graph, the normal line at that point passes through the fixed point $(1, 0)$. (The normal is the line perpendicular to the tangent line.)

a) Show that $y = y(x)$ satisfies the differential equation $y' = \dfrac{1-x}{y}$.

b) Using separation of variables, find all solutions to the differential equation. You can leave the solutions in implicit form, i.e., as equations connecting x and y.

c) Describe the curves which are their graphs. (You may have to use algebraic processes first (like completing the square) in order to change the equations into a form where you know what their graphs look like.)

Problem 8. (15) The cup of a wine-glass has the shape formed by rotating the parabola $y = x^2$ about the y-axis; its upper rim is a circle of radius 1. How much wine does it hold?

Problem 9. (10) Using the trapezoidal rule with three subdivisions ($n = 3$), estimate $\int_0^{\pi/2} \sin^2 x\, dx$. Do the work systematically, making a table of values first.

Problem 10. (15: 7,8) Let $F(x) = \int_0^x e^{-t^2} dt$.

a) Find $F'(1)$ and $F''(1)$.

b) Express $\int_1^2 e^{-u^2/4} du$ in terms of values of $F(x)$.

Problem 11. (15: 7,8) Between the two towers of a suspension bridge, each of the two main cables has the shape of the parabola $y = \frac{1}{10}x^2$ (units are kilometers). The two towers are 2 km. apart; the vertical cables from the main cable to the horizontal roadway are closely and equally spaced.

a) Set up a definite integral which gives the length of each main cable between the two towers.

b) What is the average length (to the nearest meter) of the vertical cables?

Problem 12. (20: 10,10) Evaluate

a) $\int_0^1 \frac{dx}{x^2 + 3x + 2}$; (begin by factoring the denominator). b) $\int x^2 \ln x\, dx$

Problem 13. (10) Evaluate $\int_0^1 \frac{dx}{(x^2+1)^2}$ by making the substitution $x = \tan u$; remember the limits.

Problem 14. (15) Starting at the point where $r = 1$, the point P moves counterclockwise along the polar curve $r = e^{\theta/2\pi}$, in such a way that the line segment OP makes one complete revolution. (Here O denotes the origin.)

Sketch the curve, and find the total area swept out by OP as it makes the revolution.

Problem 15. (15) Evaluate (showing work):

a) $\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x}$ b) $\lim_{x \to 1} \frac{(\ln x)^2}{x - 1}$ c) $\lim_{x \to \infty} x^2 e^{-x}$

Problem 16. (10) Evaluate $\int_1^\infty \frac{dx}{x^{3/2}}$

Problem 17. (10) For what values of p does $\sum_1^\infty \frac{n}{\sqrt{4 + n^p}}$ converge? (Indicate reasoning.)

Problem 18. (15: 10,5)

a) Find by differentiating the function $f(x) = \sqrt{1+x}$ the first four non-zero terms of its Taylor series around $x = 0$. (Show work.)

b) Use the correct answer to (a) (or your own answer, if you don't know the correct answer) to calculate $\sqrt{1.2}$ to four decimal places.

Problem 19. (10) Find the Taylor series for $\tan^{-1} x$ around $x = 0$ by using term-by-term differentiation or integration on the appropriate geometric series. Give enough terms to make the pattern clear.