

Lecture 30. December 6, 2005

Practice Problems. Course Reader: 6C-2.

1. Sequences By definition, a *sequence* of real numbers is a rule assigning to each counting number n an associated real number a_n . The integer n is called the *index* of the sequence. Usually the index begins with $n = 1$, but occasionally it begins with another integer (sometimes 0). Sequences are often specified by giving the first few values, and letting the reader infer the rule, e.g.,

$$a_1 = \frac{1}{1}, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}, \dots$$

It is always better to give a precise definition of each sequence, e.g.,

$$a_n = \frac{1}{n}, n = 1, 2, \dots$$

The most common notation for a sequence is $(a_n)_{n \geq 1}$.

A sequence $(a_n)_{n \geq 1}$ *converges to a limit* L if the sequence becomes arbitrarily close to L , and stays arbitrarily close to L . More precisely, the sequence converges to L if for every positive number ϵ , there exists an integer N (depending on the sequence and ϵ) such that for every integer $n \geq N$,

$$|a_n - L| < \epsilon.$$

In other words, the *tail* of the sequence $a_N, a_{N+1}, a_{N+2}, \dots$ are all numbers in the interval $(L - \epsilon, L + \epsilon)$. A sequence cannot have more than 1 limit: given 2 potential limits L_1 and L_2 , simply take $\epsilon = |L_1 - L_2|/2$ in the definition above. A sequence which has a limit is said to *converge*, and the limit is denoted by,

$$L = \lim_{n \rightarrow \infty} a_n.$$

A sequence which does not have a limit is said to *diverge*.

Examples.

- (i) Let L be a fixed real number. The sequence $a_n = L, n = 1, 2, \dots$ converges to L .
- (ii) The sequence $a_n = n$ diverges. In a precise sense, this sequence “diverges to ∞ ”.
- (iii) The sequence $a_n = (-1)^n$ diverges, even though it is bounded (it never gets bigger than 1 or smaller than -1).
- (iv) Let r be a real number. The sequence $a_n = r^n, n = 0, 1, 2, \dots$ converges to 0 if $|r| < 1$ and diverges if $|r| > 1$. There are 2 remaining cases. If $r = -1$, then $a_n = (-1)^n$ diverges. If $r = 1$, then $a_n = 1$ converges to 1.

2. Tests for convergence/divergence. One useful test for convergence is the *Squeezing Lemma*.

The squeezing lemma. Let $(a_n)_{n \geq 1}$, $(b_n)_{n \geq 1}$ and $(c_n)_{n \geq 1}$ be sequences such that for every index n ,

$$a_n \leq b_n \leq c_n.$$

In other words, the sequence (b_n) is “squeezed” between the sequences (a_n) and (c_n) . If (a_n) and (c_n) converge, and if,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n,$$

then also (b_n) converges and its limit equals the limit of the other 2 sequences.

Another test for convergence is the *Monotone Convergence Test*. A sequence $(a_n)_{n \geq 1}$ is called *non-decreasing* if for every index n , $a_{n+1} \geq a_n$. Similarly, a sequence (a_n) is *non-increasing* if for every index n , $a_{n+2} \leq a_n$. A sequence which is either non-decreasing or non-increasing (but not both increasing and decreasing) is called *monotone*. A sequence (a_n) is *bounded above* if there exists a real number u such that for every index n , $a_n \leq u$. The number u is an *upper bound* for the sequence. A sequence (a_n) is *bounded below* if there exists a real number l such that for every index n , $a_n \geq l$. The number l is a *lower bound* for the sequence.

Monotone Convergence Test. A non-decreasing sequence converges if and only if it is bounded above. In this case, the limit of the sequence is the least upper bound for the sequence. Similarly, a non-increasing sequence converges if and only if it is bounded below and the limit is the greatest lower bound for the sequence.

3. Series. Given a sequence $(a_n)_{n \geq 1}$, there are 2 important related sequences. The first is the *sequence of partial sums*, $(b_n)_{n \geq 1}$, defined by,

$$b_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^n a_k.$$

The second is the *sequence of partial absolute sums*, $(B_n)_{n \geq 1}$, defined by,

$$B_n = |a_1| + |a_2| + \cdots + |a_n| = \sum_{k=1}^n |a_k|.$$

If the sequence of partial sums $(b_n)_{n \geq 1}$ converges, the limit is called the *series of $(a_n)_{n \geq 1}$* , and is denoted by,

$$\sum_{k=1}^{\infty} a_k := \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k.$$

In this case it is said *the series $\sum_k a_k$ converges*. If the sequence of partial absolute sums $(B_n)_{n \geq 1}$ converges, it is said *the series $\sum_k a_k$ converges absolutely*. Although it is not obvious, if the series converges absolutely, then the series converges (this is a basic theorem from course 18.100). If a series converges but does not converge absolutely, sometimes it is said the series *converges conditionally*.

Examples. The *harmonic sequence* is the sequence $a_n = 1/n$. As will be shown soon, the *harmonic series* $\sum_n 1/n$ diverges to ∞ . The *alternating harmonic sequence* is,

$$a_n = \frac{(-1)^n}{n}.$$

The *alternating harmonic series*,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

does converge. This will also be shown soon. Since the sequence of partial absolute sums for the alternating sequence equals the sequence of partial sums for the harmonic sequence, the alternating harmonic series does not converge absolutely. It only converges conditionally.

As counter-intuitive as this might sound, the terms in the alternating harmonic series can be rearranged so that the sum converges to any real number you like! This sounds ridiculous: finite sums are independent of the order in which the summands are added, so how could this fail for

infinite sums? The answer is quite simple. Because the harmonic series $\sum_n 1/n$ diverges, the same is true for $\sum_{1/2n}$. Thus, add it up a very large number of *only* the (positive) even terms in the alternating harmonic series to make the partial sum bigger than, say, 10^6 . Now add only the first odd term $-1/2$. This has a negligible effect. Now add a large number of the remaining even terms to make the partial sum bigger than 10^7 . Now add one more odd term, $-1/3$. Continuing in this way, eventually every term in the sequence contributes to one of the partial sums. But because we add positive terms with a much higher frequency than negative terms, the sequence of partial sums is diverging to $+\infty$. Similarly, we could negative terms with a very high frequency and make the partial sums diverge to $-\infty$. Now it is not so surprising that by adding the terms in a careful order, we can make the partial sums converge to any value we like.

The pathology of the preceding paragraph occurs with any conditionally convergent series. It is a very important fact that every absolutely convergent series has only a single limit, independent of the order in which terms are added. For this reason, absolutely convergent series are much more useful than conditionally convergent series.

4. Test for convergence/divergence of series. If a series $\sum_n a_n$ converges, then the sequence (a_n) converges to 0. To see this, denote by L the limit of the sequence of partial sums (b_n) . For every positive real number ϵ , using $\epsilon/2$ in the definition of convergence of (b_n) , there exists an integer N such that for every $n \geq N$, $|b_n - L| < \epsilon/2$. But then for $n \geq N + 1$,

$$|a_n| = |b_n - b_{n-1}| = |(b_n - L) - (b_{n-1} - L)| \leq |b_n - L| + |b_{n-1} - L| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus the sequence (a_n) converges to 0. Contrapositively, if the sequence (a_n) does not converge to 0, then the series $\sum_n a_n$ diverges. This is the most basic test for divergence of a series. For example, it immediately follows that the series $\sum_{n=1}^{\infty} (-1)^n$ diverges (arguing the opposite is a favorite pasttime of “mathematical cranks”).

The most basic test for absolute convergence of a sequence follows from the monotone convergence test. The sequence of partial absolute sums,

$$B_n = \sum_{k=1}^n |a_k|,$$

is a non-decreasing sequence. Therefore, by the monotone convergence theorem, it converges if and only if it is bounded above. The most common technique for proving the sequence of partial absolute sums is bounded above is by comparing it to a larger series that is known to converge. This gives the following.

Comparison Test. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be sequences such that for every index n , $|a_n| \leq |b_n|$. If the series $\sum_{n=1}^{\infty} b_n$ converges absolutely, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

A number of common convergence tests in calculus textbooks come to nothing more than combining the comparison test with an analysis of the geometric series. Let r be a real number and let $(a_n)_{n \geq 0}$ be the geometric sequence,

$$a_n = r^n, n \geq 0,$$

(by convention, if $r = 0$, the first term a_0 is defined to be 1). By high school algebra, if $r \neq 1$, the partial sums are

$$b_n = 1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r} - \frac{1}{1 - r} r^{n+1}.$$

Observe this sequence depends on n only in the last term r^{n+1} , which is essentially the geometric sequence. Assuming $r \neq 1$, the geometric sequence r^{n+1} converges if and only if $|r| < 1$. In this case, the sequence of partial absolute sums,

$$B_n = 1 + |r| + |r|^2 + \cdots + |r|^n = \frac{1}{1 - |r|} + \frac{1}{1 - |r|} |r|^{n+1},$$

also converges. Thus, the geometric series $\sum_{n=0}^{\infty} r^n$ converges absolutely to $1/(1 - r)$ if $|r| < 1$, and diverges if $|r| > 1$ or $r = -1$. The only remaining case is when $r = 1$. Then the partial sums are $b_n = n + 1$, which diverges to ∞ . Altogether, the series $\sum_{n=0}^{\infty} r^n$ converges to $1/(1 - r)$ if $|r| < 1$, and diverges otherwise.

The ratio test. There are two tests that allow us to compare a given sequence $(a_n)_{n \geq 0}$ to a geometric sequence $(r^n)_{n \geq 1}$. If the following limit,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|,$$

exists, call it r . Then the sequence $(a_n)_{n \geq 1}$ can be compared to a sequence $(Cr^n)_{n \geq 1}$ for some choice of C . This leads to the *ratio test*: The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if the sequence $|a_{n+1}/a_n|$ converges to a real number $r < 1$ and diverges if the sequence $|a_{n+1}/a_n|$ converges to a real number $r > 1$ (in which case, the sequence $(a_n)_{n \geq 1}$ does not converge to 0). There is no information if the sequence converges to 1 or diverges.

Similarly, if the following limit,

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|},$$

exists, call it r . Then the sequence $(a_n)_{n \geq 1}$ can be compared to a sequence $(Cr^n)_{n \geq 1}$. This leads to the *root test*: The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if the sequence $\sqrt[n]{|a_n|}$ converges to a real number $r < 1$ and diverges if the sequence $\sqrt[n]{|a_n|}$ converges to a real number $r > 1$. There is no information if the sequence converges to 1 or diverges.

Comparison to an improper integral. The final test uses improper integrals to get useful information about a series. Let $(a_n)_{n \geq 1}$ be a sequence. Let $f(x) \geq 0$ be a function on $[1, \infty)$ such that for every integer n , $f(x) \geq a_n$ for all $n \leq x \leq n + 1$. If the improper integral,

$$\int_1^{\infty} f(x) dx,$$

converges, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. On the other hand, let $g(x) \geq 0$ be a function on $[1, \infty)$ such that for every integer n , $g(x) \leq a_n$ for all $n \leq x \leq n + 1$. If the improper integral,

$$\int_1^{\infty} g(x) dx,$$

diverges, then the series $\sum_{n=1}^{\infty} a_n$ does not converge absolutely. For both directions, define the sequence (c_n) by,

$$c_n = \int_n^{n+1} f(x)dx, \text{ or } c_n = \int_n^{n+1} g(x)dx.$$

The absolute partial sum of the series $\sum_{k=1}^n c_k$ is simply,

$$\sum_{k=1}^n c_k = \int_1^n f(x)dx, \text{ or } \int_1^n g(x)dx.$$

The result follows.

Examples. 1. The harmonic series. Let $(a_n)_{n \geq 1}$ be the harmonic sequence,

$$a_n = \frac{1}{n}.$$

Let $g(x)$ be the function $g(x) = 1/x$ on the interval $[1, \infty)$. Then for every integer n , $g(x) \leq a_n = 1/n$ on the interval $[n, n+1]$. By the Fundamental Theorem of Calculus, the partial sums of the sequence (c_n) are,

$$\sum_{k=1}^n c_k = \int_1^n \frac{1}{x} dx = \ln(n).$$

As n tends to ∞ , the natural logarithms $\ln(n)$ also tend to ∞ (although very slowly – $\ln(n)$ does not get bigger than a fixed real number R until n gets bigger than the much larger number e^R). Therefore the partial sums diverge. By the comparison test, the harmonic series also diverges (very slowly).

Example. 2. The Riemann zeta function. Let $s > 1$ be a real number. Define the sequence $(a_n)_{n \geq 1}$ by,

$$a_n = \frac{1}{n^s}.$$

The series $\sum_{n=1}^{\infty} 1/n^s$ equals $1 + \sum_{n=2}^{\infty} 1/n^s$, which is the same as,

$$1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^s}.$$

Let $f(x)$ be the function $f(x) = 1/x^s$. Then for each integer n , $f(x) \geq 1/(n+1)^s$ for every x in $[n, n+1]$. The partial sum of (c_n) is,

$$c_n = \int_1^n \frac{1}{x^s} dx = \left(\frac{1}{1-s} \frac{1}{x^{s-1}} \right) \Big|_1^n = \frac{1}{s-1} - \frac{1}{s-1} \frac{1}{n^{s-1}}.$$

Because s is bigger than 1, as n tends to ∞ , also n^{s-1} tends to ∞ . Therefore the partial sums tend to $1/(s-1)$. Therefore, by the comparison test, the series,

$$\sum_{n=1}^{\infty} \frac{1}{n^s},$$

converges absolutely to a value bounded by $1/(s-1)$. The value of this limit is called the *Riemann zeta function* at s , denoted

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This function is of fundamental importance in number theory. It also pops up in Fourier series and statistical mechanics. The values of $\zeta(s)$ when s is an even integer are known. The first couple are $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. There are very fundamental open problems about the Riemann zeta function. For one of these problems in particular, the Clay Mathematics Institute has offered a \$1 million prize for an accepted, refereed solution.