Lecture 25. November 17, 2005

Homework. Problem Set 7 Part I: (a)–(e)

Practice Problems. Course Reader: 5D-2, 5D-6, 5D-7, 5D-10, 5D-14

1. Inverse hyperbolic functions. There are a few other useful formulas for hyperbolic functions; for instance, the analogues of the angle-addition formulas,

$$\sinh(s+t) = \sinh(s)\cosh(t) + \cosh(s)\sinh(t),$$

$$\cosh(s+t) = \cosh(s)\cosh(t) + \sinh(s)\sinh(t).$$

These imply the double-angle formulas,

$$\sinh(2t) = 2\sinh(t)\cosh(t),$$

$$\cosh(2t) = \cosh^2(t) + \sinh^2(t) = 2\cosh^2(t) - 1 = 2\sinh^2(t) + 1.$$

From these follow the analogues of the half-angle formulas,

$$\sinh^2(t/2) = \frac{1}{2}(\cosh(t) - 1),$$

$$\cosh^2(t/2) = \frac{1}{2}(\cosh(t) + 1).$$

A beautiful feature of hyperbolic functions is that their inverse functions can be expressed in terms of simpler functions. The inverse function $\sinh^{-1}(x)$ of $\sinh(x)$ is defined on the whole real line. By definition,

$$\sinh^{-1}(x) = y \text{ if and only if } \sinh(y) = x.$$

This second equation can be written out as,

$$\frac{1}{2}(e^y - e^{-y}) = x.$$

Substituting $z = e^y$ gives,

$$\frac{1}{2}(z - z^{-1}) = x.$$

Multiplying both sides by $2z$ gives,

$$z^2 - 1 = 2xz \Leftrightarrow z^2 - 2xz - 1 = 0.$$

Completing the square gives,

$$(z-x)^2 = x^2 + 1.$$

Taking square roots gives,

$$z = x \pm \sqrt{x^2+1}.$$

Since z equals e^y, z is positive. Thus, the correct square root is,

$$z = x + \sqrt{x^2+1}.$$

Finally this gives,

$$y = \ln(z) = \ln(x + \sqrt{x^2+1}).$$

Therefore, the formula for the inverse hyperbolic sine is,

$$\sinh^{-1}(x) = \boxed{\ln(x + \sqrt{x^2+1}).}$$

The same type of argument also gives,

$$\cosh^{-1}(x) = \boxed{\ln(x + \sqrt{x^2-1}),\ x \geq 1,}$$

and

$$\tanh^{-1}(x) = \boxed{(1/2)\ln((1+x)/(1-x)),\ -1 < x < 1.}$$

2. Derivatives of the inverse hyperbolic functions. By the same methods used to compute the derivatives of inverse trigonometric functions, the derivatives of the inverse hyperbolic functions are,

$$d\sinh^{-1}(u) = \frac{du}{\sqrt{1+u^2}},$$

$$d\cosh^{-1}(u) = \frac{du}{\sqrt{u^2-1}},\ u \geq 1,$$

$$d\tanh^{-1}(u) = \frac{du}{1-u^2},\ -1 < u < 1.$$

These can also be computed using the formulas for the inverse functions.

3. Inverse substitution. The derivatives of inverse trigonometric and inverse hyperbolic functions allow us to compute more antiderivatives than before, e.g., $\int dx/(\sqrt{x^2-1})$ equals $\cosh^{-1}(x) + C$. Essentially this comes down to making a direct substitution of an inverse function, e.g., $u = \cosh^{-1}(x)$. However, this is logically equivalent to making an *inverse substition*, $x = \cosh(u)$. When the integrand is more complicated, inverse substitution is usually simpler and faster than direct substitution of an inverse function.

Example. Compute the following antiderivative,

$$\int \sqrt{a^2 - x^2}dx.$$

This is not quite the derivative of an inverse function above. However, it is clear that inverse substituting $x = a\sin(\theta)$ will simplify the integrand, because

$$a^2 - x^2 = a^2 - (a\sin(\theta))^2 = a^2(1 - \sin^2(\theta)) = a^2\cos^2(\theta).$$

Thus we have,

$$\int \sqrt{a^2 - x^2}dx, \left\{ \begin{array}{rcl} x & = & a\sin(\theta), \\ dx & = & a\cos(\theta)d\theta \end{array} \right. , \Rightarrow \int \sqrt{a^2\cos^2(\theta)}(a\cos(\theta)d\theta) = a^2 \int \cos^2(\theta)d\theta.$$

Using the half-angle formula, this becomes,

$$a^2 \int \frac{1}{2} + \frac{1}{2}\cos(2\theta)d\theta = a^2\left(\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right) + C.$$

Using the double-angle formula and back-substituting gives,

$$\int \sqrt{a^2 - x^2}dx = \boxed{(1/2)(a^2\sin^{-1}(x/a) + x\sqrt{a^2 - x^2}) + C.}$$

4. Three different kinds of integrals, three kinds of inverse substitution. The type of antiderivative where inverse substitution is most successful has the form,

$$\int \frac{F(x, \sqrt{Ax^2 + Bx + C})}{G(x, \sqrt{Ax^2 + Bx + C})}dx,$$

where A, B and C are constants, and $F(x, y)$ and $G(x, y)$ are polynomial functions in the two arguments. Inverse substitution together with partial fractions solves all such antiderivative problems.

The first step is to complete the square of the expression $Ax^2 + Bx + C$. This gives,

$$Ax^2 + Bx + C = A\left(x + \frac{B}{2A}\right)^2 - \frac{B^2 - 4AC}{4A}.$$

In particular, making the substition,

$$u = x + \frac{B}{2A}, \quad du = dx,$$

transforms the quadratic into one of 3 possible types,

$$\beta^2u^2 + \alpha^2, \beta^2u^2 - \alpha^2, -\beta^2u^2 + \alpha^2,$$

where,

$$\beta = \sqrt{|A|}, \quad \alpha = \sqrt{\frac{|B^2 - 4AC|}{|4A|}}.$$

Defining $a = \alpha/\beta$, finally the integral is transformed to one of 3 possible types,

$$\textbf{Type I:} \int \frac{F_I(u, \sqrt{a^2 - u^2})}{G_I(u, \sqrt{a^2 - u^2})}du,$$

$$\textbf{Type II:} \quad \int \frac{F_{II}(u, \sqrt{u^2 - a^2})}{G_{II}(u, \sqrt{u^2 - a^2})} du,$$

and

$$\textbf{Type III:} \quad \int \frac{F_{III}(u, \sqrt{a^2 + u^2})}{G_{III}(u, \sqrt{a^2 + u^2})} du.$$

For each of these types, there are 3 possible inverse substitutions: trigonometric, hyperbolic and rational. A flow chart of the 9 possible outcomes will be posted on the course webpage. Here are a couple of examples. In each example, the inverse rational substitution is given, although it was only briefly discussed in lecture.

Example. Compute the following antiderivative,

$$\int \frac{x^2}{\sqrt{a^2 - x^2}} dx.$$

The trigonometric inverse substition is,

$$x = a\sin(\theta), \quad dx = a\cos(\theta)d\theta.$$

The new antiderivative is,

$$\int \frac{a^2\sin^2(\theta)}{\sqrt{a^2 - a^2\sin^2(\theta)}}(a\cos(\theta)d\theta).$$

Simplifying gives,

$$\int a^2\sin^2(\theta)d\theta.$$

This can be simplified using the half-angle formula,

$$a^2\int \frac{1}{2} - \frac{1}{2}\cos(2\theta)d\theta.$$

This is easily seen to be,

$$a^2\left(\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right) + C.$$

Using the double-angle formula and back-substituting,

$$\int \frac{x^2}{\sqrt{a^2 - x^2}} dx = \boxed{(1/2)(a^2\sin^{-2}(x/a) - x\sqrt{a^2 - x^2}) + C.}$$

Alternatively, the hyperbolic inverse substitution is,

$$x = a\tanh(t), \quad dx = a\mathrm{sech}^2(t)dt.$$

The new antiderivative is,

$$\int \frac{a^2\tanh^2(t)}{\sqrt{a^2\mathrm{sech}^2(t)}} a\mathrm{sech}^2(t)dt.$$

Simplifying gives,

$$a^2\int \tanh^2(t)\mathrm{sech}(t)dt = a^2\int \frac{\sinh^2(t)}{\cosh^3(t)}dt.$$

This can be simplified a bit by multiplying numerator and denominator by $\cosh(t)$ and then expressing in terms of $\sinh(t)$ as much as possible,

$$a^2\int \frac{\sinh^2(t)}{\cosh^4(t)}\cosh(t)dt = a^2\int \frac{\sinh^2(t)}{(1+\sinh^2(t))^2}\cosh(t)dt.$$

Make the substitution $u = \sinh(t)$, $du = \cosh(t)dt$ to get,

$$a^2\int \frac{u^2}{(1+u^2)^2}du.$$

This can be rewritten as,

$$a^2\int \frac{1}{1+u^2}du - a^2\int \frac{1}{(1+u^2)^2}du.$$

The first of these terms is just $a^2\tan^{-1}(u)$. However, the second term requires another inverse substitution. All in all, this is not a very efficient approach.

Finally, the rational inverse substitution is,

$$x = a\frac{2t}{1+t^2},\ \ dx = a\frac{2(1-t^2)}{(1+t^2)^2}dt.$$

The point is that,

$$a^2 - x^2 = a^2\frac{(1-t^2)^2}{(1+t^2)^2}.$$

Thus the new antiderivative is,

$$\int \frac{4a^2t^2}{(1+t^2)^2}\frac{1+t^2}{a(1-t^2)}\frac{2a(1-t^2)}{(1+t^2)^2}dt.$$

This simplifies to,

$$8a^2\int \frac{t^2}{(1+t^2)^3}dt = 8a^2\int \frac{1}{(1+t^2)^2}dt - 8a^2\int \frac{1}{(1+t^2)^3}dt.$$

Notice, these two integrals are the same type that occurred with inverse hyperbolic substitution. But they came up more quickly: rational inverse substitution is more efficient than inverse hyperbolic substitution for this problem. However, both require a further inverse trigonometric substitution. So inverse trigonometric substitution is the most efficient for this problem.

Example. Compute the following antiderivative,

$$\int \frac{x^2}{\sqrt{x^2-a^2}}dx.$$

The trigonometric inverse substitution is,

$$x = a\sec(\theta), \quad dx = a\sec(\theta)\tan(\theta)d\theta.$$

The new antiderivative is,

$$\int \frac{a^2\sec^2(\theta)}{\sqrt{a^2\sec^2(\theta)-a^2}}a\sec(\theta)\tan(\theta)d\theta.$$

Because $\sec^2(\theta)-1$ equals $\tan^2(\theta)$, simplifying gives,

$$a^2\int \sec^3(\theta)d\theta = a^2\int \frac{1}{\cos^3(\theta)}d\theta.$$

This can be simplifed by multiplying numerator and denominator by $\cos(\theta)$ and then expressing in terms of $\sin(\theta)$ as much as possible,

$$a^2\int \frac{1}{\cos^4(\theta)}\cos(\theta)d\theta = a^2\int \frac{1}{(1-\sin^2(\theta))^2}\cos(\theta)d\theta.$$

Make the substitution $u = \sin(\theta)$, $du = \cos(\theta)d\theta)$ to get,

$$a^2\int \frac{1}{(1-u^2)^2}du.$$

This can be computed using partial fractions (not yet discussed).

Alternatively, the hyperbolic inverse substitution is,

$$x = a\cosh(t), \quad dx = a\sinh(t)dt.$$

The new antiderivative is,

$$\int \frac{a^2\cosh^2(t)}{\sqrt{a^2\cosh^2(t)-a^2}}a\sinh(t)dt.$$

Since $\cosh^2(t)-1$ equals $\sinh^2(t)$, simplifying gives,

$$a^2\int \cosh^2(t)dt.$$

This can be simplified using the analogue of the half-angle formula,

$$a^2\int \frac{1}{2}+\frac{1}{2}\cosh(2t)dt.$$

This is easily seen to be,
$$a^2\left(\frac{t}{2}-\frac{1}{4}\sinh(2t)\right)+C.$$
Using the double-angle formula and back-substituting,
$$\int \frac{x^2}{\sqrt{x^2-a^2}}dx = \frac{1}{2}\left(a^2\cosh^{-1}(x/a)-x\sqrt{x^2-a^2}\right).$$
Using the formula for $\cosh^{-1}(x/a)$, this becomes,
$$\int \frac{x^2}{\sqrt{x^2-a^2}}dx = \boxed{(1/2)(a^2\ln(x+\sqrt{x^2-a^2})-x\sqrt{x^2-a^2})+C.}$$

Finally, the rational substitution is,
$$x = a\frac{1+t^2}{2t},\quad dx = a\frac{-(1-t^2)}{2t^2}dt.$$
The point is that,
$$a^2-x^2 = a^2\frac{(1-t^2)^2}{(2t)^2}.$$
Thus the new antiderivative is,
$$\int \frac{a^2(1+t^2)^2}{4t^2}\frac{2t}{a(1-t^2)}\frac{-a(1-t^2)}{2t^2}dt.$$
This simplifies to,
$$-\frac{a^2}{4}\int\frac{(1+t^2)^2}{t^3}dt = -\frac{a^2}{4}\int\frac{1}{t^3}+\frac{2}{t}+tdt.$$
This evaluates to,
$$-\frac{a^2}{4}\left(\frac{-1}{2t^2}+2\ln(t)+\frac{t}{2}\right)+C.$$
This is clearly the easiest of the 3 methods for computing the antiderivative, for this problem. However, there still remains the formidable problem of solving for $t=t(x)$, back-substituting, and simplifying the resulting expression. All in all, inverse hyperbolic substitution is the most efficient for this problem.