18.01 Calculus
Due by 2:00pm sharp
Friday, Oct. 14, 2005

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Solutions to Problem Set 3

Part I/Part II

Part I(20 points)

(a)(2 points)	p.119,	Section 4.1,	Problem 11
(b)(2 points)	p. 119,	Section 4.1,	Problem 24
(c)(2 points)	p. 122,	Section 4.2,	Problem 11
(d)(2 points)	p. 129,	Section 4.3,	Problem 28
(e)(2 points)	p. 137,	Section 4.4,	Problem 7
(f)(2 points)	p. 137,	Section 4.4,	Problem 28
(g)(2 points)	p. 142,	Section 4.5,	Problem 8
(h)(2 points)	p. 142,	Section 4.5,	Problem 21
(i)(2 points)	p. 146,	Section 4.6,	Problem 2(a)
(j)(2 points)	p. 146,	Section 4.6,	Problem 2(b)

Solution (a) Since $y = x + x^{-1}$, then $y' = 1 - x^{-2}$. Therefore

$$\begin{array}{cccccccccccccc} & & & x=-1 & & & x=0 & & & x=1 & & \\ y & \nearrow & \nearrow & & \searrow & \searrow & & \searrow & \searrow & & \nearrow & \nearrow \\ y' & + & + & 0 & - & - & & - & - & 0 & + & + \end{array}$$

We infer that there is a local maximum at $x = -1$, where $y = -2$, and a local minimum at $x = 1$ where $y = 2$. Considering that $y = f(x)$ almost equals x for large values of x, we are now ready to sketch its graph, which is shown in Figure 1.

Solution (b) Let them intersect at the point (x_0, y_0). Because they intersect at (x_0, y_0)

$$\sin ax_0 = \cos ax_0$$

This implies that $\tan ax_0 = 1$ which in turn implies $ax_0 = \pi/4 + n\pi$ for some integer n. Therefore $\sin ax_0 = \cos ax_0 = \pm 1/\sqrt{2}$. Moreover, because the curves intersect at right angles

$$\frac{d}{dx}(\sin ax)|_{x_0} \cdot \frac{d}{dx}(\cos ax)|_{x_0} = -1$$

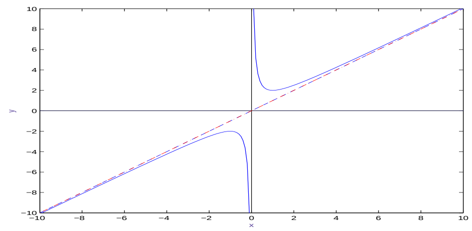


Figure 1: Solution to (a): The graph of $y = x + \frac{1}{x}$.

in other words

$$a \cos ax_0 . (-a \sin ax_0) = -1$$

or $\cos ax_0 \sin ax_0 = \left(\pm 1/\sqrt{2}\right)^2 = 1/a^2$, hence $\boxed{a = \sqrt{2}}$ since $a > 0$.

Solution (c) Since $y = \frac{4x^2}{x^2+3} = 4 - \frac{12}{x^2+3}$, then by chain rule

$$y' = \frac{24x}{(x^2+3)^2}$$

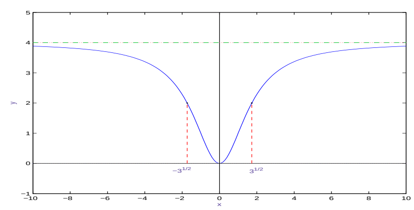
which, when combined with the quotient rule, gives

$$y'' = \frac{24(x^2+3)^2 - 24(2)(2)x^2(x^2+3)}{(x^2+3)^4} = \frac{24(-3x^2+3)}{(x^2+3)^3}$$

We see that $y'' = 0$ for $-3x^2 + 3 = 0$, i.e $x = \pm 1$.Noting that $y = 4$ is a horizontal asymptote which y gets arbitrarily close to as $x \to \pm\infty$, we form the table below

	$x = -\infty$			$x = -1$			$x = 0$			$x = 1$			$x = \infty$
y	4	$\searrow$	$\searrow$	1	$\searrow$	$\searrow$	0	$\nearrow$	$\nearrow$	1	$\nearrow$	$\nearrow$	4
y'	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$
y''	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	0	$-$	$-$	$-$
y		*concave*	*up*			*concave*	*down*				*concave*	*up*	

We're now ready to sketch the graph as in Figure



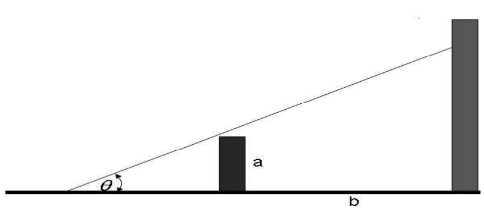


Figure 2: The description of the wall and ladder as in problem (d).

Solution (d)

The length of the ladder is given by

$$l = \frac{a}{\sin\theta} + \frac{b}{\cos\theta} \tag{1}$$

Then, we let

$$\frac{dl}{d\theta} = -a\frac{\cos\theta}{\sin^2\theta} + b\frac{\sin\theta}{\cos^2\theta}$$

equal to zero, which gives the equation for the critical angle θ :

$$a\frac{\cos\theta}{\sin^2\theta} = b\frac{\sin\theta}{\cos^2\theta}$$

i.e.

$$\tan\theta = \sqrt[3]{\frac{a}{b}}$$

$$\sin\theta = \sqrt[3]{\frac{a}{b}}(1+\sqrt[3]{\frac{a^2}{b^2}})^{-1}, \quad \cos\theta = (1+\sqrt[3]{\frac{a^2}{b^2}})^{-1}$$

Putting back into (1), we obtain

$$\boxed{l_{\min} = (1+\sqrt[3]{\tfrac{a^2}{b^2}})(a^{2/3}b^{1/3}+b) = (a^{2/3}+b^{2/3})^2}$$

Solution (e) The weight of the sheet of the metal is proportional to its area, which is

$$A = \pi r^2 + 2\pi r h \tag{2}$$

where r is the radius of the base and h the height. We want to maximize the volume of the can:

$$V = \pi r^2 h$$

We express h from the (2)

$$h = \frac{A - \pi r^2}{2\pi r}$$

therefore

$$V = \frac{r}{2}(A - \pi r^2)$$

By differentiating

$$\frac{dV}{dr} = \frac{1}{2}[A - \pi r^2 - 2\pi r^2] = 0$$

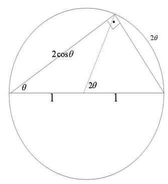


Figure 3: A sketch of the circular pool in problem (d).

from which we obtain

$$r = \sqrt{\frac{A}{3\pi}} \implies h = \frac{A - A/3}{2\pi\sqrt{\frac{A}{3\pi}}} = \sqrt{\frac{A}{3\pi}} = r$$

therefore the required ratio is

$$\boxed{ratio = \frac{height}{diameter} = \frac{h}{2r} = \frac{1}{2}}$$

Solution (f)

The time spent is

$$T = \frac{2\cos\theta}{3} + \frac{2\theta}{6}$$

Differentiating and equating it to zero

$$\frac{dT}{d\theta} = -\frac{2}{3}\sin\theta + \frac{1}{3} = 0 \ \Rightarrow \sin\theta = \frac{1}{2}$$

hence $\theta = \pi/6 = 30^0$.

Solution (g) The length of the string is given by

$$L = \sqrt{80^2 + x^2}$$

where x is the horizontal distance between the boy and the kite.

Differentiating with respect to time, we obtain

$$\frac{dL}{dt} = \frac{dL}{dx}\frac{dx}{dt} = \frac{x}{\sqrt{80^2 + x^2}}\frac{dx}{dt}$$

We observe that $x = 60$ when $L = 100$. Putting this and the given $\frac{dx}{dt} = 20$ ft/s in the last equation yields

$$\boxed{\frac{dL}{dt} = \frac{60}{100} 20 ft/s = 12 ft/s}$$

which is the speed of the string let out by the boy.

Solution (h) We are given that the ice ball melts proportional to its area, in symbols

$$\frac{dV}{dt} = -kA$$

where $V = \frac{4}{3}\pi r^3$ is the volume and $A = 4\pi r^2$ is the area of the ice ball with radius r. Rewriting the above equation and using the chain rule

$$\frac{d}{dt}(\frac{4}{3}\pi r^3) = 4\pi r^2 \frac{dr}{dt} = -k4\pi r^2$$

we obtain

$$\frac{dr}{dt} = -k$$

therefore

$$r = r_0 - kt \tag{3}$$

Now, at half the volume the radius is given by

$$\frac{4}{3}\pi r^3 = \frac{1}{2}\frac{4}{3}\pi r_0^3 \;\Rightarrow\; r = 2^{-1/3} r_0$$

The half volume is reached at $t = 2$, therefore using (3)

$$2^{-1/3} r_0 = r_0 - 2k \Rightarrow k = \frac{1}{2} r_0 (1 - 2^{-1/3})$$

The radius r hits zero at time

$$t = \frac{r_0}{k} = \frac{2}{1 - 2^{-1/3}}$$

Therefore, the additional time necessary to melt is,

$$\boxed{2/(\sqrt[3]{2} - 1) \text{ hours.}}$$

Solution (i) We are given $y = f(x) = x^3 + 3x^2 - 6$. Then

$$y' = f'(x) = 3x^2 + 6x = 3x(x + 2)$$

We observe that $f'(x) < 0$ only for $-2 < x < 0$, otherwise $f'(x) \geq 0$.

Some of its values are: $f(-2) = -2$, $f(-1) = -4$. $f(0) = -6$, $f(1) = -2$, $f(2) = 14$.

Now, since $f(-2) < 0$, and $f'(x) > 0$ for $x < -2$, we can say that f has no roots for $x < -2$.Also,

since $f(-2) < 0$, and $f'(x) < 0$ for $-2 < x < 0$, we can say that f has no roots for $-2 < x < 0$.Moreover, since $f(0) = -6$, and $f'(x) > 0$ for $x > 0$, we can say that f has at most one root for

$x > 0$.

Furthermore, since $f(1) = -2 < 0$ and $f(2) = 14 > 0$, we conclude that f has exactly one root which is between 1 and 2 by the intermediate value theorem.

Now, we are ready to calculate the Newton's iterates

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 3x_n^2 - 6}{3x_n^2 + 6x_n}$$

with the initial guess $x_1 = 1$. Then

$$x_2 = x_1 - \frac{x_1^3 + 3x_1^2 - 6}{3x_1^2 + 6x_1} = 1 - \frac{-2}{9} = \frac{11}{9} = 1.222222222$$

Repeating yields

$$x_3 = x_2 - \frac{x_2^3 + 3x_2^2 - 6}{3x_2^2 + 6x_2} = 1.19621502380123$$
$$x_4 = 1.19582343356330$$
$$\boxed{x_5 = 1.195823}$$

Since x_4 and x_5 have the first the six digits the same, we conclude that x_5 is at least accurate to six digits.

Solution (j) We are given $y = f(x) = x^3 + 3x - 8$. Then

$$y' = f'(x) = 3x^2 + 3 > 0 \text{ always}$$

Therefore $y = f(x)$ is always increasing, which implies it cannot have more than one root.

Since $f(2) = 6 > 0$, and $f(1) = 4 < 0$, by the intermediate value theorem, f has a root between 2 and 1. Now, we are ready to calculate the Newton's iterates

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 3x_n^2 - 8}{3x_n^2 + 3}$$

with the initial guess $x_1 = 1$. Then

$$x_2 = x_1 - \frac{x_1^3 + 3x_1 - 8}{3x_1^2 + 3} = 1 + \frac{4}{6} = 1.66666666666667$$

Repeating yields

$$\begin{aligned} x_3 &= x_2 - \frac{x_2^3 + 3x_2^2 - 8}{3x_2^2 + 3} = 1.52287581699346 \\ x_4 &= 1.51279230892909 \\ x_5 &= 1.51274532763374 \\ &\boxed{x_6 = 1.512745} \end{aligned}$$

Since x_5 and x_6 have the first the six digits the same, we conclude that x_6 is at least accurate to six digits.

Part II(30 points)

Problem 1(10 points) A function $f(x)$ is defined to be,

$$f(x) = \begin{cases} (1+\sin(x))/\cos(x) & \text{if } \cos(x) \neq 0, \\ 0 & \text{if } \cos(x) = 0. \end{cases}$$

For $-2\pi \leq x \leq 2\pi$, sketch the graph of $y = f(x)$. Do each of the following.

(i) Label all vertical asymptotes. Use the form "y = number".

(ii) Label all local maxima and local minima (if any). Give the coordinates for each labelled point.

(iii) Label all inflection points (if any). Give the coordinates and the derivative of each labelled point.

(iv) Label each region where the graph is concave up. Label each region where the graph is concave down.

Warning: This graph is trickier than it seems! Before attempting the problem, it may be helpful to use a computer or graphing calculator to get an idea what the graph looks like.

Solution to Problem 1

Since the function $y = f(x)$ is periodic with period 2π, it suffices to do the analysis for only $[0, 2\pi]$.

Solution to (i) The vertical asymptotes MAY be located at places where $\cos x = 0$, which happens for $x = \frac{\pi}{2}, \frac{3\pi}{2}$ in $[0, 2\pi]$.

Since $1 + \sin x \neq 0$ for $x = \frac{3\pi}{2}$, this is a vertical asymptote. On the other hand, $1 + \sin x = 0$ for $x = \frac{\pi}{2}$, so this point requires further analysis. We use the good old L'hopitals rule to conclude

$$\lim_{x \to \frac{\pi}{2}} \frac{1 + \sin x}{\cos x} = \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{-\sin x} = 0$$

therefore there is no asymptote at $x = \frac{\pi}{2}$ where the function is continuous.

Solution to (ii) By the quotient rule,

$$f'(x) = \frac{\cos^2 x + (1+\sin x)\sin x}{\cos^2 x} = \frac{1+\sin x}{\cos^2 x} = \frac{1+\sin x}{1-\sin^2 x} = \frac{1}{1-\sin x}$$

which is always non-negative. Therefore f is nondecreasing and there are no local extrema.

Solution to (iii) By the chain rule

$$f''(x) = \frac{\cos x}{(1-\sin x)^2}$$

which changes sign whenever $\cos x = 0$ which happens at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. However, there is a vertical asymptote at $x = \frac{\pi}{2}$, therefore the only inflection point occurs when $x = \frac{3\pi}{2}$. The coordinates of this point is given by $(\frac{3\pi}{2}, 0)$. The slope at the inflection point is $f'(\frac{3\pi}{2}) = 1/2$.

Solution to (iv) The function is concave up if $\cos x > 0$ and concave down if $\cos x < 0$. The table below summarizes all we have mentioned:

$x =$	0			$\pi/2$			π			$3\pi/2$			2π
y	1	$\nearrow$	$\nearrow$	$\pm\infty$	$\nearrow$	$\nearrow$	-1	$\nearrow$	$\nearrow$	0	$\nearrow$	$\nearrow$	$\nearrow$
		concave	up		concave	down					concave	up	
y'	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
y''	$+$	$+$	$+$	0	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$

Now we're ready to draw the graph of $y = f(x)$. We first draw the part for $0 \leq x \leq 2\pi$, and then extend it periodically to $-2\pi \leq x \leq 2\pi$, see Figure 4.

Problem 2(10 points) Figure 5 depicts two fixed rays L and M meeting at a fixed acute angle ϕ. A line segment of fixed length c is allowed to slide with one endpoint on line L and one endpoint on line M. Denote by a the distance from the origin of ray L to the endpoint of the segment on line L. Denote by b the distance from the origin of ray M to the endpoint of the segment on line M. Denote by θ the angle made by the line segment and the ray L at the point where they meet.

For the position of the line segment making b maximal, express a, b and θ in terms of ϕ. Show your work. You may find the *law of cosines* useful,

$$c^2 = a^2 + b^2 - 2ab\cos(\phi).$$

Solution to Problem 2 We start with the law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\phi \tag{4}$$

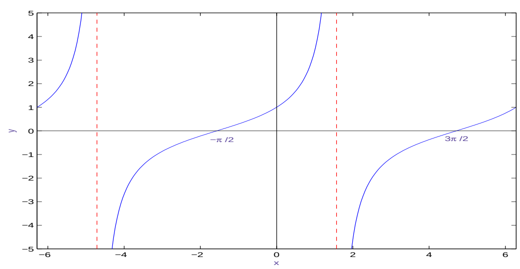


Figure 4: The graph of the function $y = \frac{1+\sin x}{\cos x}$

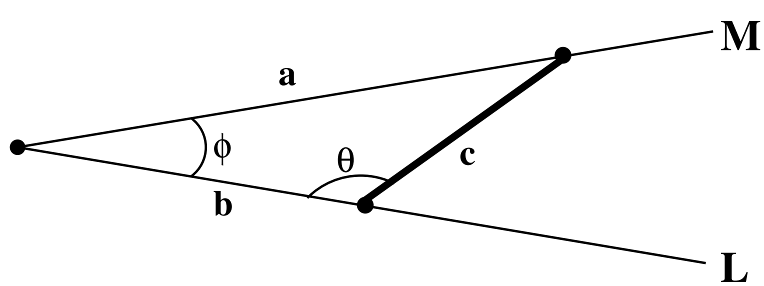


Figure 5: Sliding line segment of length c

We can consider b as a function of a : in fact, there are two such functions, however, their maximums are the same and they take on their maximum at the same point.

Implicitly differentiating (4) with respect to a, we obtain

$$0 = 2a + 2b\frac{db}{da} - 2b\cos\phi - 2a\frac{db}{da}\cos\phi$$

At the maximum of $b = b(a)$, we know that $\frac{db}{da} = 0$. Therefore

$$0 = 2a - 2b\cos\phi \implies a = b\cos\phi$$

Plugging this back into (4), we obtain

$$c^2 = b^2(\cos^2\phi + 1) - 2b^2\cos\phi = b^2(1 - \sin^2\phi) = b^2\sin^2\phi$$

therefore the maximum b is

$$\boxed{b = \frac{c}{\sin\phi}} \tag{5}$$

which is assumed for

$$\boxed{a = b\cos\phi = c\cot\phi}$$

We're also required to find an expression for θ when b is maximum. We call α the angle corresponding to the edge b in the abc triangle, then by sines theorem

$$\frac{b}{\sin\alpha} = \frac{c}{\sin\phi}$$

But this, combined with (5), implies that $\sin\alpha = 1$, and hence $\alpha = \pi/2$. From the relation $\phi + \theta + \alpha = \pi$, we find out that

$$\boxed{\theta = \frac{\pi}{2} - \phi}$$

in the case when b is maximal.

Problem 3(10 points) Solve Problem 17 from §4.4, p. 137 of the textbook. You are free to use any (valid) method you like. You may find the following remarks useful.

Figure 6 depicts an isosceles triangle circumscribed about a circle of radius R. The two similar sides each have length $A + B$, and the third side has length $2A$. Express the area of each right triangle in terms of either $\tan(\alpha)$ or $\tan(\beta)$. Because the sum of the angles of a circle is 2π, β equals $\pi - 2\alpha$. Recall the double-angle formula for tangents,

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - (\tan(\theta))^2},$$

and the complementary angle formula for tangents,

$$\tan(\pi - \theta) = -\tan(\theta).$$

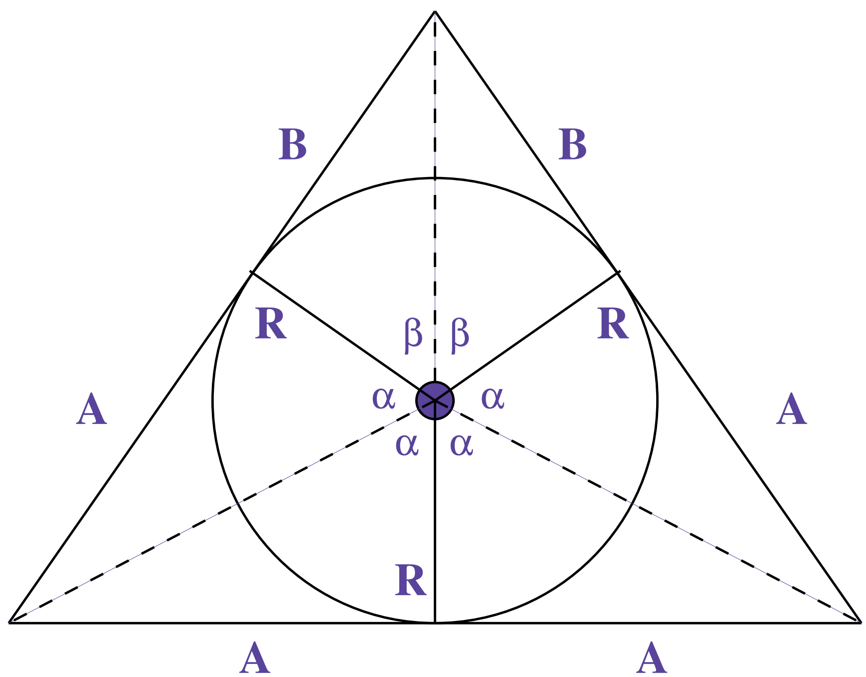


Figure 6: An isosceles triangle circumscribing a circle of radius R

Using these, express $\tan(\beta)$, and thus the total area of the triangle, in terms of $T = \tan(\alpha)$. Now minimize this expression with respect to T, find the corresponding angle α, and the height of the triangle.

Solution to Problem 3 The total area can be written down as

$$T = \frac{1}{2}R(4A + 2B)$$

but

$$A = \tan\alpha, B = R\tan\beta = R\tan(\pi - 2\alpha) = -R\tan 2\alpha$$

Note that $\tan 2\alpha < 0$ (and hence $\pi/2 < 2\alpha$). Rewriting the total area

$$\begin{aligned} T &= \frac{1}{2}R(4\tan\alpha - 2R\tan 2\alpha) = R^2(2\tan\alpha - \tan 2\alpha) \\ &= R^2(2\tan\alpha - \frac{2\tan\alpha}{1-\tan^2\alpha}) \\ &= R^2(2u - \frac{2u}{1-u^2}) \end{aligned}$$

where $u = \tan\alpha$. In order to find the extrema of this last quantity, all we need to do is to differentiate with respect to u and set the result equal to zero:

$$\begin{aligned}\frac{d}{du}R^2(2u - \frac{2u}{1+u^2}) &= R^2(2 - \frac{2(1-u^2)+4u^2}{(1-u^2)^2}) = 0 \\ &\Rightarrow 2 - \frac{2(1+u^2)}{(1-u^2)^2} = 0 \\ &\Rightarrow \frac{(1+u^2)}{(1-u^2)^2} = 1\end{aligned}$$

hence

$$(1-u^2)^2 - u^2 - 1 = u^4 - 3u^2 = u^2(u^2-3) = 0$$

therefore $u^2 = 0$ or $u^2 = 3$. Recalling that $u = \tan\alpha$, which needs to be a positive number for the area to be maximum, we conclude

$$u = \tan\alpha = \sqrt{3}$$

hence

$$\boxed{\alpha = \frac{\pi}{3} = 60^o}$$