3.35 – Fracture and Fatigue
Problem Set 6 – Solutions
November 25, 2003

(a) In order to calculate the stress intensity factors at the tip of a kinked crack, one must use a polar coordinate system to evaluate the normal (hoop) stress and the shear stress, which can then be used to calculate k_1 and k_2, respectively. Please see B. Cotterell and J.R. Rice, Int. J. Frac., Vol. 16 (1980) and M.L. Williams, J. Appl. Mech., Vol 24(1), p. 109-114 (1957) for full details.

(b) K_I and K_{II} denote the global mode I and mode II stress intensity factors. The problem assumes global mode I loading only, so $K_{II} = 0$:

$\therefore$ From Equation (9.116), $k_1 \approx a_{11}(\alpha) K_I$

$$k_2 \approx a_{21}(\alpha) K_I$$

Using triple angle formulas, the expressions for a_{11} and a_{21} in Equation (9.117) can be reduced as follows:

Let $x = \frac{\alpha}{2}$: $a_{11}(\alpha) = \frac{1}{4}(3\cos x + \cos 3x)$

$\cos 3x = 4\cos^3 x - 3\cos x \quad \therefore a_{11}(\alpha) = \frac{1}{4}(3\cos x + 4\cos^3 x - 3\cos x)$

$$a_{11}(\alpha) = \cos^3\left(\frac{\alpha}{2}\right)$$

$$a_{21}(\alpha) = \frac{1}{4}(\sin x + \sin 3x)$$

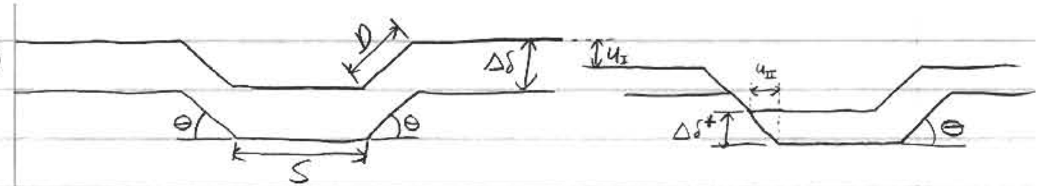
$\sin 3x = 3\sin x - 4\sin^3 x \quad \therefore a_{21}(\alpha) = \frac{1}{4}(\sin x + 3\sin x - 4\sin^3 x)$

$$= \sin x\,(1 - \sin^2 x)$$

$$a_{21}(\alpha) = \sin\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right)$$

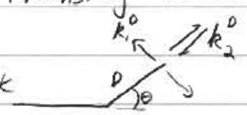
$$\boxed{\begin{aligned} \therefore k_1 &= \cos^3\left(\frac{\alpha}{2}\right) K_I \\ k_2 &= \sin\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right) K_I \end{aligned}}$$

(C)



Local stress intensity factors:

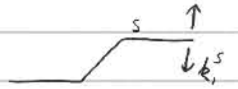
kinked crack



$k_1^D \approx \left(\cos^3 \frac{\theta}{2}\right) K_I$

$k_2^D \approx \left(\sin\frac{\theta}{2} \cos^2 \frac{\theta}{2}\right) K_I$

straight crack



$k_1^S = K_I$

$k_2^S = 0$

We know that $k^2 = k_1^2 + k_2^2 \rightarrow k = (k_1^2 + k_2^2)^{1/2}$

$\therefore\ k^S = (K_I^2 + 0)^{1/2} = K_I$

$k^D = \left(\left[\cos^3\frac{\theta}{2} K_I\right]^2 + \left[\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} K_I\right]^2\right)^{1/2}$

$= \left(\left[\cos^3\frac{\theta}{2}\right]^2 + \left[\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right]^2\right)^{1/2} K_I$

$= \left(\cos^6\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\cos^4\left(\frac{\theta}{2}\right)\right)^{1/2} K_I$

$= \left[\cos^4\left(\frac{\theta}{2}\right)\left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\right)\right]^{1/2} K_I$

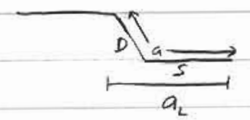
$k^D = \cos^2\left(\frac{\theta}{2}\right) K_I$

Taking the weighted average of the effective stress intensity factors:

$$\bar{k} = \frac{K_I S + \cos^2\left(\frac{\theta}{2}\right) K_I D}{S + D} = \left(\frac{D\cos^2\left(\frac{\theta}{2}\right) + S}{D + S}\right) K_I$$

$$\boxed{\overline{\Delta k} = \left(\frac{D\cos^2\left(\frac{\theta}{2}\right) + S}{D + S}\right) \Delta K_I} \rightarrow 14.22$$

(d) The effective rate of fatigue crack propagation will also be reduced:



$a = D + S \rightarrow$ Total crack growth

$a_L = D\cos\theta + S \rightarrow$ Effective crack growth

$$\frac{a_L}{a} = \frac{D\cos\theta + S}{D+S} \qquad a_L = \left(\frac{D\cos\theta + S}{D+S}\right) a$$

If $\left(\frac{da}{dN}\right)_L$ represents the growth rate of a linear crack

$$\boxed{\left(\frac{da}{dN}\right)_{effective} = \left(\frac{D\cos\theta + S}{D+S}\right)\left(\frac{da}{dN}\right)_L} \rightarrow \text{Eqn } 14.23$$

(e) The value of ΔK_{eff} will be further reduced by fracture surface mismatch:

$\Delta\delta^* = u_{II}\tan\theta$

$\Delta\delta = \Delta\delta^* + u_I = u_{II}\tan\theta + u_I$

$\left.\begin{array}{l}\Delta K_{c1} \text{ varies with } \sqrt{\Delta\delta^*} \\ \Delta K_I \text{ varies with } \sqrt{\Delta\delta}\end{array}\right\} \quad \frac{\Delta K_{c1}}{\Delta K_I} = \sqrt{\frac{\Delta\delta^*}{\Delta\delta}} = \sqrt{\frac{u_{II}\tan\theta}{u_{II}\tan\theta + u_I}}$

$$= \sqrt{\frac{\frac{u_{II}}{u_I}\tan\theta}{\frac{u_{II}\tan\theta + u_I}{u_I}}}$$

$$= \sqrt{\frac{\chi\tan\theta}{\chi\tan\theta + 1}} \quad \text{where } \chi = \frac{u_{II}}{u_I}$$

Note: $\Delta K_{eff} = \left(\frac{D\cos^2(\frac{\theta}{2}) + S}{D+S}\right)\left(\Delta K_I - \Delta K_{c1}\right)$

$$\frac{\Delta K_{eff}}{\Delta K_I} = \left(\frac{D\cos^2(\frac{\theta}{2}) + S}{D+S}\right)\left(1 - \frac{\Delta K_{c1}}{\Delta K_I}\right)$$

$$= \left(\frac{D\cos^2(\frac{\theta}{2}) + S}{D+S}\right)\left(1 - \sqrt{\frac{\chi\tan\theta}{1+\chi\tan\theta}}\right)$$

Inverting:

$$\boxed{\frac{\Delta K_I}{\Delta K_{eff}} = \left(\frac{D\cos^2(\frac{\theta}{2}) + S}{D+S}\right)^{-1}\left(1 - \sqrt{\frac{\chi\tan\theta}{1+\chi\tan\theta}}\right)^{-1}} \rightarrow \text{Eqn } 14.24$$

(f) For k_1 and k_2 to be meaningful in describing the strength of the singularity ahead of a deflected crack, the plastic zone size must be small, relative to the kink length (i.e. it must be well within the zone of K-dominance). When the plastic zone size becomes large, the use of k_1 and k_2 is no longer valid. As noted in Suresh and Shih (1986), the plastic zone size under mixed mode loading is much larger than that associated with pure mode I loading, under the same load amplitude. This should also be accounted for when analyzing deflected cracks.

Crack tip plasticity can further enhance the effects of crack deflection (i.e. increase in fracture initiation toughness and crack growth resistance). Plasticity effectively reduces the tensile stress at the crack tip. See Suresh and Shih, Figure 13.

Problem 2:

I found that $D = 0.255mm$, $S = 0.418mm$, $\theta = 60°$

We must first reduce the linear crack growth rate:

$$\left(\frac{da}{dN}\right)_{effective} = \left(\frac{D\cos\theta + S}{D+S}\right)\left(\frac{da}{dN}\right)_L = 0.81\left(\frac{da}{dN}\right)_L$$

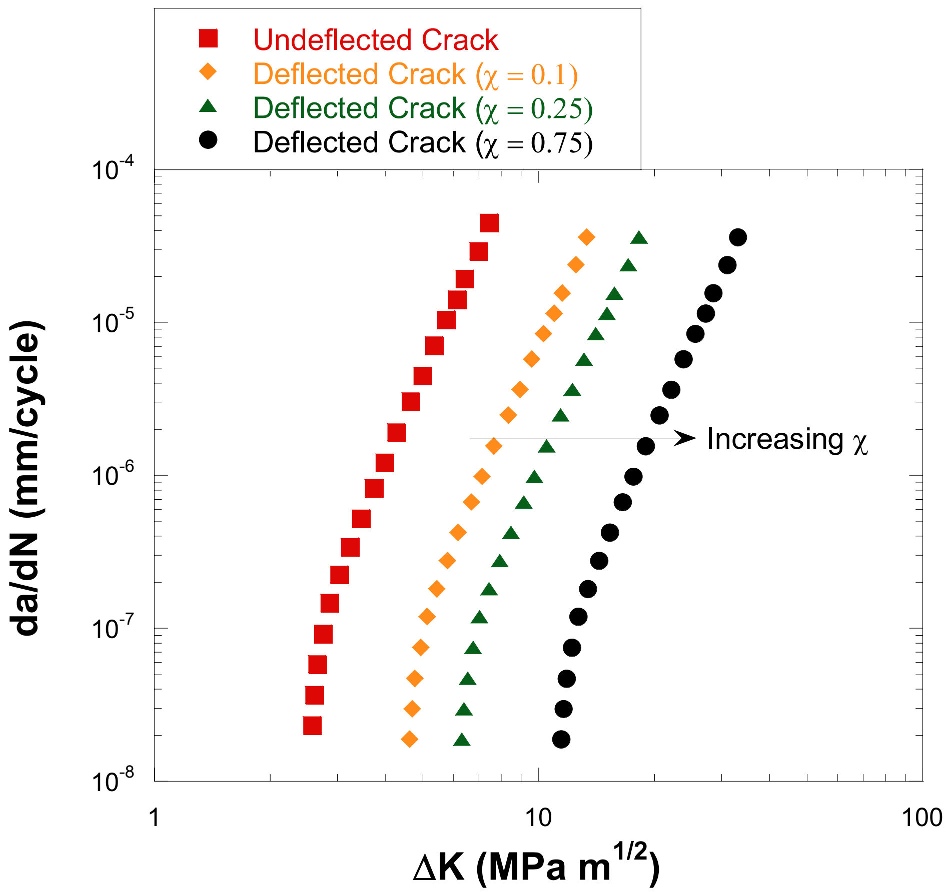
Then we must adjust the apparent stress intensity factor range:

$$\left(\frac{D\cos^2\left(\frac{\theta}{2}\right) + S}{D+S}\right)^{-1}\left(1 - \sqrt{\frac{\chi\tan\theta}{1+\chi\tan\theta}}\right)^{-1} = 1.795 \quad \text{for } \chi = 0.1$$

$$= 2.457 \quad \text{for } \chi = 0.25$$

$$= 4.444 \quad \text{for } \chi = 0.75$$

See the plot below


Undeflected Crack
Deflected Crack ($\chi = 0.1$)
Deflected Crack ($\chi = 0.25$)
Deflected Crack ($\chi = 0.75$)
Increasing χ
da/dN (mm/cycle)
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
1
10
100
ΔK (MPa $m^{1/2}$)