Department of Materials Science and Engineering
Massachusetts Institute of Technology
3.044 Materials Processing – Spring 2013

Exam I

Wednesday, March 13, 2013

The Rules:

1) No books allowed; no computers allowed; etc.

2) A simple calculator is allowed

3) One hand written 3x5 index card may be used as a crutch

4) Complete 5 out of the 6 problems. If you do more than 5 problems, I will grade the first 5 that are not crossed out.

5) Make sure that you READ THE QUESTIONS CAREFULLY

6) Reasonable assumptions are permitted, but please state them clearly.

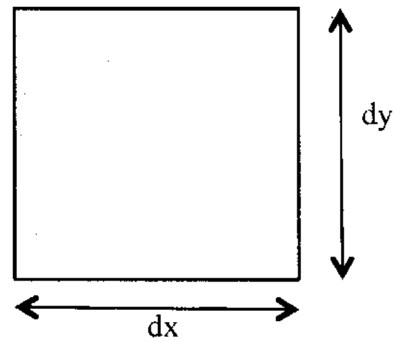
7) Supplementary materials are attached to the end of the test (eqns., etc.)

8) WRITE YOUR NAME HERE:

SOLUTIONS

Problem #1: Why Not Start at the Beginning, with a Heat Balance Problem?

Let's derive the heat conduction equation, but with some bells and whistles added. Let's add anisotropy!



Consider a small square (2D) patch as shown; we will work the problem in 2D and assume a unit thickness on the Z axis, which we will call L_z. Unfortunately, this material is anisotropic and has two relevant conductivities, k_x and k_y on the two axes.

Part A: Write down the "word" equation for the heat balance in this system.

$$\left(\text{HEAT IN } X - \text{HEAT OUT } X\right) + \left(\text{HEAT IN } Y - \text{HEAT OUT } Y\right) = \text{HEAT ACCUMULATED}$$

Part B: Replace the words with mathematical symbols, but do not rearrange yet.

$$dy\,L_z\left(q_{IN_x} - q_{OUT_x}\right) + dx\,L_z\left(q_{IN_y} - q_{OUT_y}\right) = \rho c_p \frac{\partial T}{\partial t}\,dx\,dy\,L_z$$

$$dy\,L_z\left(k_x \left.\frac{\partial T}{\partial x}\right|_{x+dx} - k_x \left.\frac{\partial T}{\partial x}\right|_{x}\right) + dx\,L_z\left(k_y \left.\frac{\partial T}{\partial y}\right|_{y+dy} - k_y \left.\frac{\partial T}{\partial y}\right|_{y}\right) = \rho c_p \frac{\partial T}{\partial t}\,dx\,dy\,L_z$$

Part C: Simplify to the greatest extent possible, to provide a differential equation that is the conduction equation for a 2-D anisotropic plate!

$$k_x \frac{\left(\left.\frac{\partial T}{\partial x}\right|_{x+dx} - \left.\frac{\partial T}{\partial x}\right|_{x}\right)}{dx} + k_y \frac{\left(\left.\frac{\partial T}{\partial y}\right|_{y+dy} - \left.\frac{\partial T}{\partial y}\right|_{y}\right)}{dy} = \rho c_p \frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial t} = \frac{k_x}{\rho c_p}\frac{\partial^2 T}{\partial x^2} + \frac{k_y}{\rho c_p}\frac{\partial^2 T}{\partial y^2}$$

Part D: Show that you can reproduce the standard, isotropic form of the 2D conduction equation in the limiting case where $k_x = k_y$

$k = k_x = k_y$, THEN
$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

Problem #2: Hot and Bothered, Cooled and Bored

Take a small block (L = 0.5 cm) of a refractory material like Al_2O_3, and consider how it cools from a very high temperature (1500° C) while sitting in air.

Part A) On its way from high temperatures to low, we have a competition between radiation, convection, and conduction. At what temperatures is each of these dominant?

$$Bi = \frac{hL}{k \leftarrow \sim 6\ W/mK} = 0.008$$ ⇒ NO GRADIENTS IN SOLID EVER. ∴ CONDUCTION NEVER LIMITING.

RADIATION VS. CONVECTION:

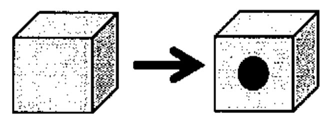
$$\text{"}Q\text{"} = \frac{q_{conv}}{q_{rad}} = \frac{h(T_s - T_E)}{\varepsilon\sigma(T_s^4 - T_E^4)} \approx \frac{h}{\varepsilon\sigma\Delta T^3} =$$

ε ~0.5 FOR Al_2O_3

AT $\Delta T = 1500°$, $Q = 0.10$ (RAD DOMINANT)
AT $\Delta T = 1000°$, $Q = 0.35$ } BOTH MATTER
AT $\Delta T = 500°$, $Q = 2.8$
AT $\Delta T = 300°$, $Q = 13$ (CONV DOMINANT)

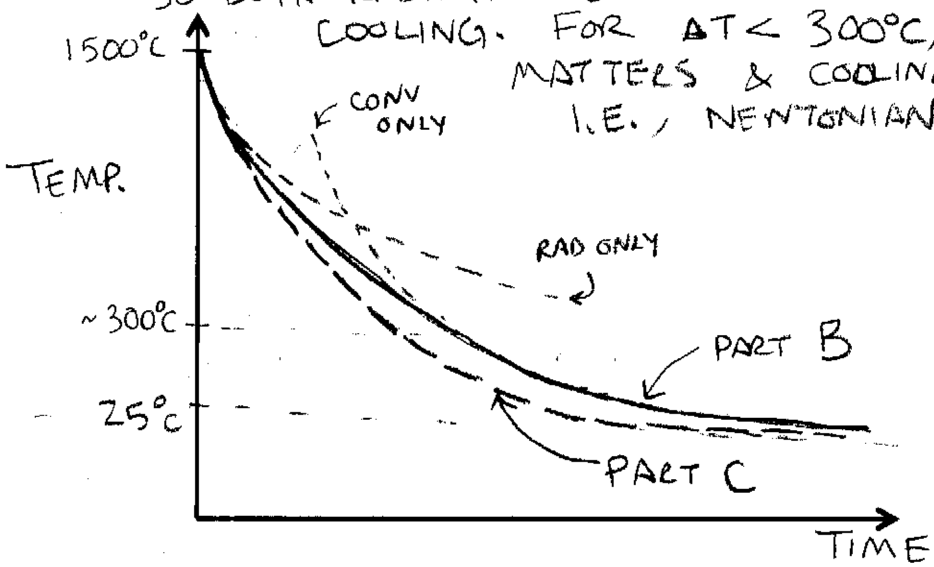
Part B) On the axes below, draw a graph showing schematically how the temperature at the surface of the block drops with time. Explain in one sentence what you have drawn.

Part C) Now a small bore-hole is made through the center of the block along one axis. Draw a second curve on your graph, showing how this hole affects the cooling of the block. Again, explain in one sentence or so what your thoughts are that led you to this drawing.



→ FOR RADIATION "NEWTONIAN" COOLING, $T \propto t^{-1/3}$. (DOESN'T MATTER WHETHER YOU KNOW THIS, IT JUST SHOULD CLEARLY BE SOME KIND OF DECREASING POWER LAW THING)

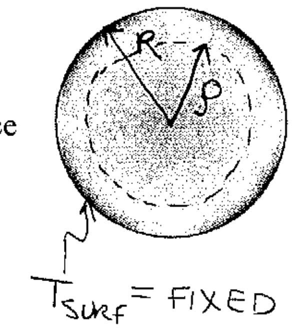
B) INITIALLY, THE TEMPERATURE DROPS AT THE RATE DICTATED BY COOLING VIA RADIATION ONLY. HOWEVER, THE "Q" NUMBER IS BETWEEN 0.1 AND 10 FOR ~300° < ΔT < 1500°, SO BOTH RADIATION & CONVECTION CONTRIBUTE, ACCELERATING THE COOLING. FOR ΔT < 300°C, RADIATION NO LONGER MATTERS & COOLING IS EXPONENTIAL W/TIME, I.E., NEWTONIAN.



C) THE COOLING BY RADIATION IS UNAFFECTED BY THE BOREHOLE. HOWEVER, THE CONVECTIVE COOLING IS ACCELERATED BY THE BOREHOLE BECAUSE IT INCREASED THE SURFACE AREA.

Problem #3: Know When to Neglect Something

We've discussed in some detail when you can neglect something—now please apply those principles to a case we all know well.



Consider heat conduction into the surface of a cylinder with a fixed surface temperature. If we focus on the earliest stages of the process, and the penetration of the heat is small, then we all know intuitively that we can treat this like a 1-D Cartesian problem. Let's try to formalize this a bit more.

Part A) Write down the proper form of the conduction equation for this geometry, and also write down the assumed form for a 1-D Cartesian problem.

PROPER FORM:

1 $$\frac{\partial T}{\partial t} = \alpha \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

CARTESIAN FORM:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Part B) Compare the two forms of the diffusion equation, and identify the terms that need to be neglected in order for the problem to become 1-D Cartesian

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{r}\left(\frac{\partial T}{\partial r} + r\frac{\partial^2 T}{\partial r^2}\right) = \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}$$

2 IF $\frac{\partial^2 T}{\partial r^2} >> \frac{1}{r}\frac{\partial T}{\partial r}$, THEN THE CARTESIAN APROXIMATION IS ACCEPTABLE. I.E., WE MUST NEGLECT THE $\boxed{\frac{1}{r}\frac{\partial T}{\partial r}}$ TERM.

Part C) Write down a mathematical condition under which you would consider it is OK to neglect the cylindrical geometry in favor of the 1-D Cartesian one. If there is any assumption here, please state it explicitly.

WE NEED $\dfrac{\frac{\partial^2 T}{\partial r^2}}{\frac{1}{r}\frac{\partial T}{\partial r}} \geq 10$ TO JUSTIFY THE APPROXIMATION.

2 ∂T CAN BE APPROXIMATED AS $(T_{SURF} - T_{INITIAL})$, AND ∂r CAN BE APPROXIMATED AS $(R-\rho)$, AND $\frac{1}{r} = \frac{1}{R}$, SO THE RATIO BECOMES:

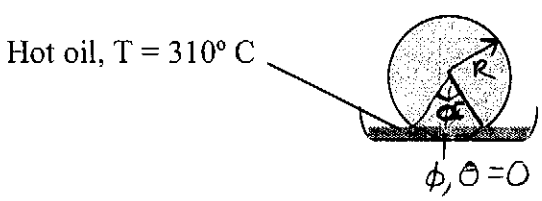
$$\frac{\frac{\Delta T}{(R-\rho)^2}}{\frac{1}{R}\frac{\Delta T}{(R-\rho)}} \geq 10 \rightarrow \frac{R}{R-\rho} \geq 10 \rightarrow \rho \leq \frac{9}{10}R,$$

WHERE ρ IS THE DEPTH OF PENETRATION OF HEAT INTO THE CYLINDER.

Problem #4: Meet Rudiger, the Incompetent Heat Transfer Engineer

On your first day on the job at a major manufacturing concern, you are introduced to Rudiger, the engineer in charge of designing annealing and heat treatment furnaces.

Rudiger is considering the heat treatment of a spherical anode of copper. The goal of the exercise is to heat the entire ball to at least 300° C to relieve stresses in it. Rudiger places the ball in a shallow pool of hot oil.



Rudiger reasons that in hot oil at 310° C, the ball will heat up to above 300° C. He says, "For a spherical geometry, the steady state condition is a uniform temperature". He allows enough time that a steady state is achieved.

Part A:

Qualitatively explain the flaw in Rudiger's logic.

Part B:

Write down a proper set of equations that define this problem, including the equation we must solve, and the boundary conditions for steady state. (Don't solve it, just write the equations)

A) A UNIFORM TEMPERATURE IS THE STEADY STATE FOR A SPHERE IN A UNIFORM ENVIRONMENT. THIS CASE DOES NOT HAVE UNIFORM BOUNDARY CONDITIONS.

B) SOLVE $$0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial T}{\partial \phi}\right)$$

WITH THE BOUNDARY CONDITIONS:

SUFFICIENT

- FOR $r = R,\ \phi \le \frac{\alpha}{2}$, $q_{\text{IN TO SPHERE}}\Big|_{r=R,\ \phi\le\alpha/2} = h_{OIL}\left(T_{SURF}\Big|_{r=R,\ \phi\le\frac{\alpha}{2}} - T_{OIL}\right)$
- FOR $r = R,\ \phi > \frac{\alpha}{2}$ $q_{\text{IN TO SPHERE}}\Big|_{r=R,\ \phi>\frac{\alpha}{2}} = h_{AIR}\left(T_{SURF}\Big|_{r=R,\ \phi>\frac{\alpha}{2}} - T_{AIR}\right)$

OR, YOU CAN USE THIS B.C.:

- SYMMETRY ABOUT THE $\phi = 0$ AXIS:
  - WHERE $\phi = 0,\ 0 \le r < R$, & • $\phi = 180°,\ 0 \le r < R$, NO GRADIENTS, $\frac{\partial T}{\partial(\text{DIRECTION} \perp \text{TO } \phi\text{-AXIS})} = 0$

ONLY NEED A B.C.

Problem #5: Enlighten Me

Each of the five statements below is false (or at best, is not always true!). Make a simple but nontrivial change (for example, deleting a few words, adding a few words, or both) to each sentence to make it generally true. Write 1 sentence of explanation for each to back up your changes.

A) When heating an object by immersing it in hot gas or a hot fluid, a higher M number AND BIOT NUMBER is desirable to obtain the most rapid kinetics.

THE M NUMBER ONLY TELLS YOU ABOUT THE SPEED OF KINETICS VIA RADIATION. THE BIOT NUMBER TELLS YOU ABOUT THE SPEED OF KINETICS VIA CONVECTION. IDEALLY, MAKE BOTH LARGE TO MAXIMIZE SPEED.

B) Cubes, spheres, and any other 3D shape of the same material and characteristic dimension AND WITH $Bi < 0.1$ will cool at the same rate given the same initial and boundary conditions.

THE COOLING RATE IS INDEPENDANT OF GEOMETRY ONLY WITHIN THE NEWTONIAN REGIME. OTHERWISE, THEY WILL ALL COOL DIFFERENTLY.

C) When cooling an object, a ~~lower~~ HIGHER Biot number FOR THE SAME OBJECT means that ~~conduction~~ CONVECTION is faster, and therefore cooling is faster.

FOR A GIVEN OBJECT, IF THE BIOT NUMBER INCREASES, THEN CONVECTION IS ACCELERATED, AND COOLING IS ACCELERATED. HOWEVER, THE BIOT NUMBER TELLS YOU NOTHING WHEN COMPARRING THE COOLING OF DIFFERENT OBJECTS.

D) Two objects of the same ~~volume~~ CHARACTERISTIC LENGTH and the same material AND THE SAME ENVIRONMENT hold the same total amount of heat, and therefore cool at the same rate under Newtonian conditions.

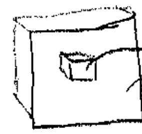
THE NEWTONIAN COOLING RATE DEPENDS ON ρc_p (SAME IF SAME MATERIAL), L (NOT VOLUME), AND h (SAME IF IN THE SAME ENVIRONMENT).

E) If conduction is rapid then both M and Bi ~~will be~~ ARE low, and both radiation and convection ~~will~~ MAY be important. SWITCH

SWITCH THE CAUSALITY: IF M & Bi ARE LOW, THEN THERE ARE NO GRADIENTS IN THE SOLID, AND BOTH RADIATION & CONVECTION MIGHT BE IMPORTANT, THOUGH YOU DO NOT KNOW WHETHER BOTH OR JUST ONE ARE DOMINANT UNTIL YOU COMPARE $h\Delta T$ & $\varepsilon\sigma(T_{surf}^4 - T_{source}^4)$.

Problem #6: Wherein a Block of Steel Meets a Fate Common to Lobsters

A cubic centimeter block of steel is at room temperature, and suddenly immersed in 10 cm^3 of water at 100 C.



Part A:

What is the steady state for this system?

STEADY STATE IS A UNIFORM TEMPERATURE EVERYWHERE

1 IN THE H_2O - STEEL SYSTEM. THAT TEMPERATURE IS:

WEIGHTED AVERAGE

$$\left\{ \frac{(100°C)(10\ cm^3) + (20°C)(1\ cm^3)}{\underbrace{(11\ cm^3)}_{\text{SIZE OF SYSTEM}}} \right. = 93°C = \text{STEADY-STATE TEMPERATURE}$$

(ASSUMING THE SYSTEM IS INSULATED)

Part B:

Write out a global heat balance for the system. Explain how this could be used to ascertain the steady state (but no need to plug in numbers, etc.).

2

$$q_{\text{PUT INTO CUBE BY WATER}} = q_{\text{ACCUMULATED IN THE CUBE}}$$

TOTAL SURF. AREA OF CUBE, $=6L^2$

$$A\,h\left(T_{cube}\big|_{surface} - T_{water}\right) = V\rho c_p \frac{\partial T_{cube}}{\partial t}$$

$$Bi = \frac{hL}{k} = \frac{(1400\ \frac{W}{m^2K})(0.005m)}{(35\ W/mK)} = 0.2$$

$$-q_{\text{PUT INTO CUBE BY WATER}} = q_{\text{ACCUMULATED IN THE WATER}}$$

$$-Ah\left(T_{cube}\big|_{surf} - T_{water}\right) = V\rho c_p \frac{\partial T_{water}}{\partial t}$$

THESE ARE COUPLED EQNS FOR T_{cube} & T_{surf}.

Part C: T_{water} & T_{cube} ARE BOTH FUNCTIONS OF TIME

It may be difficult to calculate just how long it would take to get to the steady state in this case. Maybe instead we can calculate lower and upper bounds. Please explain how you would obtain such bounds in this case.

2 LOWER BOUND: ASSUME $T_{surf} = T_{water}$, I.E., THE BIOT NUMBER IS HIGH. THIS IS THE FASTEST WAY TO WARM THE CUBE.

UPPER BOUND: ASSUME T_{cube} IS SPATIALLY UNIFORM, I.E., THE BIOT NUMBER IS SMALL. THIS IS THE SLOWEST WAY TO WARM THE CUBE.