Nov. 14 2005: Lecture 21:

Higher-Order Ordinary Differential Equations

Reading:
Kreyszig Sections: §2.1 (pp:54–70) , §2.2 (pp:72–75) , §2.3 (pp:76–80)

Higher-Order Equations: Background

For first-order ordinary differential equations (ODEs), $F(y'(x), y(x), x)$, one value $y(x_o)$ was needed to specify a particular solution. For second-order equations, two independent values are needed. This is illustrated in the following forward-differencing example.

MATHEMATICA® Example: Lecture-21

A Second-Order Forward Differencing Example

Recall the example in Lecture 19 of a first-order differencing scheme: at each iteration the function grew proportionally to its current size. In the limit of very small forward differences, the scheme converged to exponential growth.
Now consider a situation in which function's current rate of growth increases proportionally to two terms: its current rate of growth and its size.

Change in Value's Rate of Change $+\,\alpha$ (the Value) $+\,\beta$ (Value's Rate of Change) $= 0$

To Calculate a forward differencing scheme for this case, let Δ be the forward-differencing increment.

$$\left(\frac{\frac{F_{i+2}-F_{i+1}}{\Delta} - \frac{F_{i+2}-F_{i+1}}{\Delta}}{\Delta} \right) + \alpha F_i + \beta \left(\frac{F_{i+1} - F_i}{\Delta} \right) = 0$$

and then solve for the "next increment" F_{i+2} if F_{i+1} and F_i are known.

Forward Difference Formulae

Linear Differential Equations; Superposition in the Homogeneous Case ....

A linear differential equation is one for which the function and its derivatives are each linear—that is they appear in distinct terms and only to the first power. In the case of a homogeneous linear differential equation, the solutions are *superposable*. In other words, sums of solutions and their multiples are also solutions.

Therefore, a linear heterogeneous ordinary differential equation can be written as a product of general functions of the dependent variable and the derivatives for the n-order linear case:

$$\begin{aligned} 0 &= f_0(x) + f_1(x)\frac{dy}{dx} + f_2(x)\frac{d^2y}{dx^2} + \cdots + f_n(x)\frac{d^ny}{dx^n} \\ &= (f_0(x), f_1(x), f_2(x), \ldots, f_n(x)) \cdot \left(1, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) \\ &= \vec{f}(x) \cdot \vec{D_n} y \end{aligned} \tag{21-1}$$

The homogeneous n^{th}-order linear ordinary differential equation is defined by $f_0(x) = 0$ in Eq. 21-1:

$$\begin{aligned} 0 &= f_1(x)\frac{dy}{dx} + f_2(x)\frac{d^2y}{dx^2} + \cdots + f_n(x)\frac{d^ny}{dx^n} \\ &= (0, f_1(x), f_2(x), \cdots, f_n(x)) \cdot \left(1, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) \\ &= \vec{f_{hom}}(x) \cdot \vec{D_n} y \end{aligned} \tag{21-2}$$

Equation 21-1 can always be multiplied by $1/f_n(x)$ to generate the general form:

$$\begin{aligned} 0 &= F_0(x) + F_1(x)\frac{dy}{dx} + F_2(x)\frac{d^2y}{dx^2} + \cdots + \frac{d^ny}{dx^n} \\ &= (F_0(x), F_1(x), F_2(x), \ldots, 1)) \cdot (1, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}) \\ &= \vec{F}(x) \cdot \vec{D_n} y \end{aligned} \tag{21-3}$$

For the second-order linear ODE, the heterogeneous form can always be written as:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x) \tag{21-4}$$

and the homogeneous second-order linear ODE is:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{21-5}$$

Basis Solutions for the homogeneous second-order linear ODE ....

Because two values must be specified for each solution to a second order equation—the solution can be broken into two basic parts, each deriving from a different constant. These two independent solutions form a *basis pair* for any other solution to the homogeneous second-order linear ODE that derives from any other pair of specified values.

The idea is the following: suppose the solution to Eq. 21-5 is found the particular case of specified parameters $y(x = x_0) = A_0$ and $y(x = x_1) = A_1$, the solution $y(x; A_0, A_1)$ can be written as the sum of solutions to two *other problems*.

$$y(x; A_0, A_1) = y(x, A_0, 0) + y(x, 0, A_1) = y_1(x) + y_2(x) \tag{21-6}$$

where

$$\begin{aligned} y(x_0, A_0, 0) = A_0 \quad &\text{and} \quad y(x_1, A_0, 0) = 0 \\ y(x_0, 0, A_1) = 0 \quad &\text{and} \quad y(x_1, 0, A_1) = A_1 \end{aligned} \tag{21-7}$$

from these two solutions, any others can be generated.

The two arbitrary integration constants can be included in the definition of the general solution:

$$\begin{aligned} y(x) &= C_1 y_1(x) + C_1 y_2(x) \\ &= (C_1, C_2) \cdot (y_1, y_2) \end{aligned} \tag{21-8}$$

Second Order ODEs with Constant Coefficients

The most simple case—but one that results from models of many physical phenomena—is that functions in the homogeneous second-order linear ODE (Eq. 21-5) are constants:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \tag{21-9}$$

If two independent solutions can be obtained, then any solution can be formed from this basis pair.

Surmising solutions seems a sensible strategy, certainly for shrewd solution seekers. Suppose the solution is of the form $y(x) = \exp(\lambda x)$ and put it into Eq. 21-9:

$$(a\lambda^2 + b\lambda + c)e^{\lambda x} = 0 \tag{21-10}$$

which has solutions when and only when the quadratic equation $a\lambda^2 + \lambda x + c = 0$ has solutions for λ.

Because two solutions are needed and because the quadratic equation yields two solutions:

$$\begin{aligned} \lambda_+ &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ \lambda_- &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \end{aligned} \tag{21-11}$$

or by removing the redundant coefficient by diving through by a:

$$\begin{aligned} \lambda_+ &= \frac{-\beta}{2} + \sqrt{(\frac{\beta}{2})^2 - \gamma} \\ \lambda_- &= \frac{-\beta}{2} - \sqrt{(\frac{\beta}{2})^2 - \gamma} \end{aligned} \tag{21-12}$$

where $\beta \equiv b/a$ and $\gamma \equiv c/a$.

Therefore, any solution to Eq. 21-9 can be written as

$$y(x) = C_+ e^{\lambda_+ x} + C_- e^{\lambda_- x} \tag{21-13}$$

MATHEMATICA® Example: Lecture-21

Solutions to $\frac{d^2y}{dx^2} + \beta\frac{dy}{dx} + \gamma y = 0$

Because the fundamental solution depend on only two parameters β and γ, the behavior of all solutions can be visualized in the γ-β plane.

1. Insert $y(x) = \exp(\lambda x)$ into the ODE $y'' + \beta y' + \gamma y = 0$ and solve for a condition on λ that solutions exist (assuming real coefficients γ and β).

2. Plot the condition that the roots are complex in the γ-β plane. This is the region of parameter space that gives oscillatory solutions (because $\exp(r + \imath\theta) = \exp(r)(\cos(x) + \imath \sin(x))$)

3. Plot the conditions that the λ are real—these are the monotonically growing ($\lambda > 0$) or shrinking ($\lambda < 0$) solutions

4. Plot the conditions that the real part is negative, this is the damped oscillatory region.

5. Plot the conditions that the real part is positive, this is the unbounded growth region.

6. Use the MATHEMATICA® function Reduce to find the three regions: ($\lambda_+ > 0, \lambda_- > 0$)—monotonically growing solutions, ($\lambda_+ > 0, \lambda_- < 0$)—one growing and one decaying solution, ($\lambda_+ < 0, \lambda_- < 0$)—monotonically decaying solutions.

The behavior of all solutions can be collected into a simple picture:

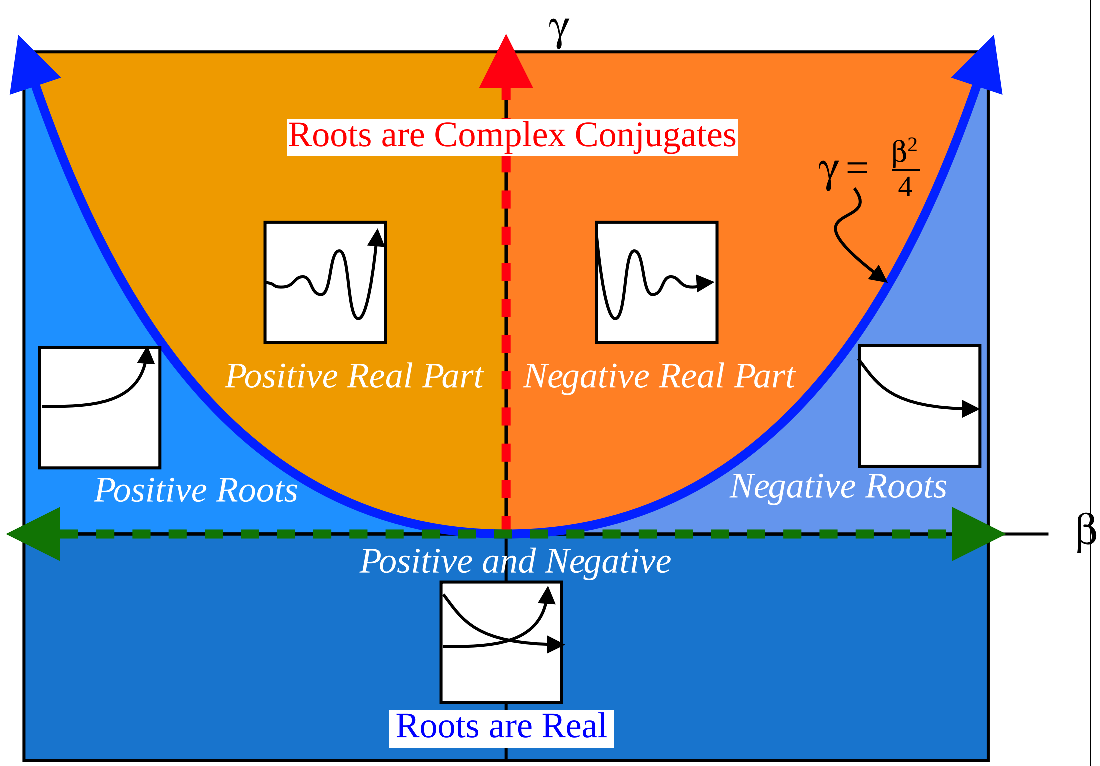

Figure 21-1: The behaviors of the linear homogeneous second-order ordinary differential equation $\frac{d^2y}{dx^2} + \beta\frac{dy}{dx} + \gamma y = 0$ plotted according the behavior of the solutions for all β and γ.

The case that separates the complex solutions from the real solutions, $\gamma = (\beta/2)^2$ must be treated separately, for the case $\gamma = (\beta/2)^2$ it can be shown that $y(x) = \exp(\beta x/2)$ and $y(x) = x\exp(\beta x/2)$ form an independent basis pair (see Kreyszig *AEM*, p. 74).

Boundary Value Problems

It has been shown that all solutions to $\frac{d^2y}{dx^2} + \beta\frac{dy}{dx} + \gamma y = 0$ can be determined from a linear combination of the basis solution. Disregard for a moment whether the solution is complex or real, and ignoring the special case $\gamma = (\beta/2)^2$. The solution to any problem is given by

$$y(x) = C_+ e^{\lambda_+ x} + C_- e^{\lambda_- x} \tag{21-14}$$

How is a solution found for a particular problem? Recall that *two values* must be specified to get a solution—these two values are just enough so that the two constants C_+ and C_- can be obtained.

In many physical problems, these two conditions appear at the boundary of the domain. A typical problem is posed like this:

Solve

$$m\frac{d^2y(x)}{dx^2} + \nu\frac{dy(x)}{dx} + ky(x) = 0 \qquad \text{on } 0 < x < L \tag{21-15}$$

subject to the boundary conditions

$$y(x = 0) = 0 \qquad \text{and} \qquad y(x = L) = 1$$

or, solve

$$m\frac{d^2y(x)}{dx^2} + \nu\frac{dy(x)}{dx} + ky(x) = 0 \qquad \text{on } 0 < x < \infty \tag{21-16}$$

subject to the boundary conditions

$$y(x = 0) = 1 \qquad \text{and} \qquad y'(x = L) = 0$$

When the value of the function is specified at a point, these are called *Dirichlet* conditions; when the derivative is specified, the boundary condition is called a *Neumann* condition. It is possible have boundary conditions that are mixtures of Dirichlet and Neumann.

MATHEMATICA® Example: Lecture-21

Determining Solution Constants from Boundary Values

1. Using Solve, find the specific solution to $y(0) = 0$ and $y(l) = 1$.
2. Using Solve, find the specific solution to $y(0) = 1$ and $y'(0) = 0$.

When the domain is infinite or semi-infinite and the physical situation indicates that the solution must be bounded, then one can automatically set the constants associated with roots with real positive parts to zero, because these solutions grow without bound.

Fourth Order ODEs, Elastic Beams

Another linear ODE that has important applications in materials science is that for the deflection of a beam. The beam deflection $y(x)$ is a linear fourth-order ODE:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y(x)}{dx^2}\right) = w(x) \tag{21-17}$$

where $w(x)$ is the load density (force per unit length of beam), E is Young's modulus of elasticity for the beam, and I is the moment of inertia of the cross section of the beam:

$$I = \int_{\mathrm{A}_{\times-sect}} y^2 dA \tag{21-18}$$

is the second-moment of the distribution of heights across the area.

If the moment of inertia and the Young's modulus do not depend on the position in the beam (the case for a uniform beam of homogeneous material), then the beam equation becomes:

$$EI\frac{d^4y(x)}{dx^4} = w(x) \tag{21-19}$$

The homogeneous solution can be obtained by inspection—it is a general cubic equation $y_{homog}(x) = C_0 + C_1x + C_2x^2 + C_3x^3$ which has the four constants that are expected from a fourth-order ODE.

The particular solution can be obtained by integrating $w(x)$ four times—if the constants of integration are included then the particular solution naturally contains the homogeneous solution.

The load density can be discontinuous or it can contain Dirac-delta functions $F_o\delta(x - x_o)$ representing a point load F_o applied at $x = x_o$.

It remains to determine the constants from boundary conditions. The boundary conditions can be determined because each derivative of $y(x)$ has a specific meaning as illustrated in Fig. 23-3.

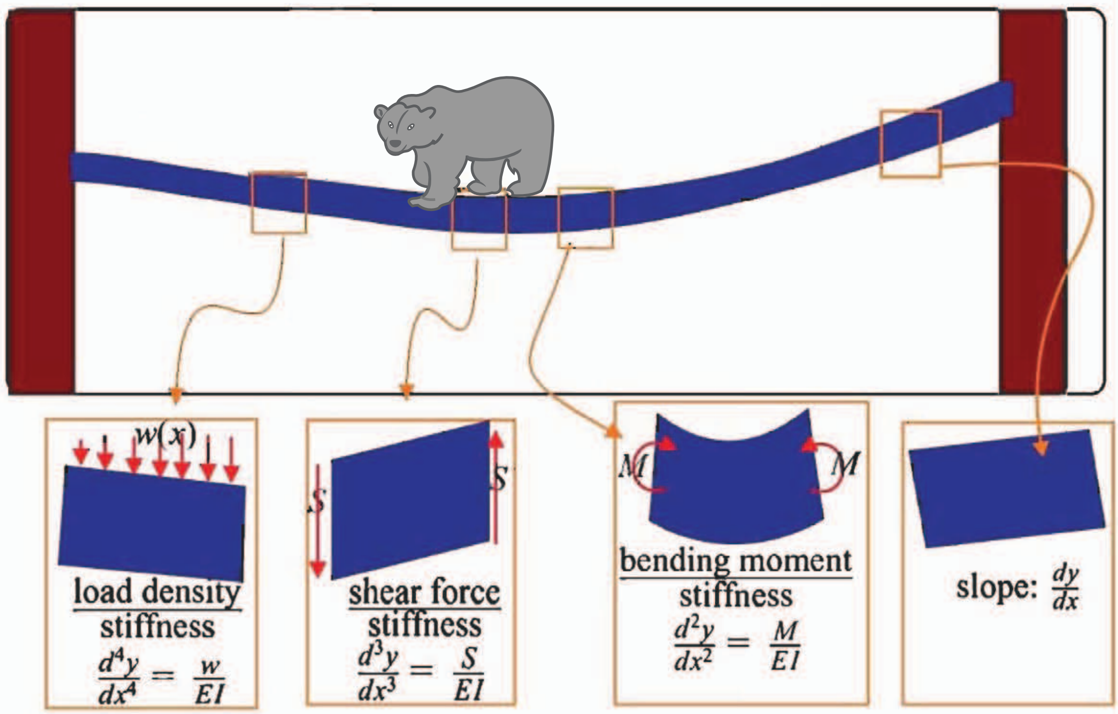

Figure 21-2: The shape of a loaded beam is determined by the loads applied over its length and its boundary conditions. The beam curvature is related to the local moment (imagine two handles rotated in opposite directions on a free beam) divided by the effective beam stiffness. Shear forces are related to the rate of change of moment along the beam.

There are common loading conditions that determine boundary conditions:

Free No applied moments or applied shearing force:

$$M = \left.\frac{d^2y}{dx^2}\right|_{boundary} = 0$$
$$S = \left.\frac{d^3y}{dx^3}\right|_{boundary} = 0$$

Point Loaded local applied moment, displacement specified.

$$M = \left.\frac{d^2y}{dx^2}\right|_{boundary} = M_o$$

$$y(x)|_{boundary} = y_o$$

Clamped Displacement specified, slope specified

$$\left.\frac{dy}{dx}\right|_{boundary} = s_o$$

$$y(x)|_{boundary} = y_o$$

MATHEMATICA® Example: Lecture-23

Visualizing beam deflections

general solutions to beam equation