MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Departments of Electrical Engineering, Mechanical Engineering, and the Harvard-MIT Division of Health Sciences and Technology

6.022J/2.792J/BEH.371J/HST542J: Quantitative Physiology: Organ Transport Systems

PROBLEM SET 6

SOLUTIONS

April 1, 2004

Problem 1

A pressure wave, P_i, incident on an arterial (or bronchial) bifurcation will suffer a reflection, P_r, of magnitude

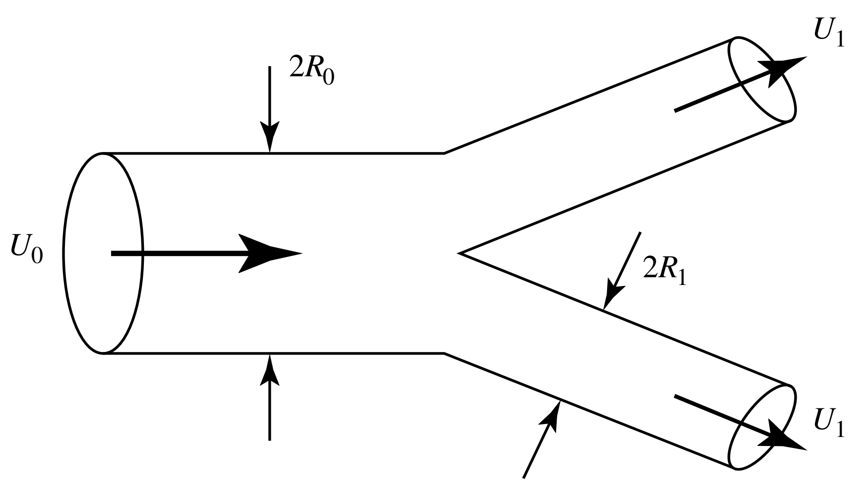
$$\frac{P_r}{P_i} \equiv \Gamma = \frac{Z_L - Z_o}{Z_L + Z_o}$$

where Z_o is the upstream arterial impedance, and Z_L is the impedance of the bifurcation. The impedance, Z, of a vessel of radius R may be calculated from the formulæderived in the notes:

$$Z^2 = \frac{\rho}{AC_u} \qquad C_u \approx \frac{2\pi R^3}{hE}$$

where ρ is the fluid density, A is the vessel cross-section, h is the vessel wall thickness, and E is the vessel modulus of elasticity.

Figure 1:



A. Assuming a symmetric bifurcation, ρ = constant, $E_0 = E_1$, $\frac{h_1}{R_1} = \frac{h_0}{R_0}$, and no increase in total cross-sectional area across the bifurcation, calculate the reflection coefficient.

$$\begin{aligned} C_u &\approx \frac{2\pi R^3}{hE} \\ A &= \pi R^2 \\ \frac{1}{Z_L} &= \frac{1}{Z_l} + \frac{1}{Z_l} \\ &= \frac{2}{Z_l} \\ \rho &= same \end{aligned}$$

$$
\begin{aligned}
Z^2 &= \frac{\rho}{AC_u} = \frac{\rho h E}{\pi R^2 2\pi R^3} \\
&= \frac{\rho h E}{2\pi^2 R^5} \\
\frac{P_r}{P_i} &\equiv \Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{\frac{1}{2}\left(\frac{h_1 E_1}{R_1^5}\right)^{1/2} - \left(\frac{h_0 E_0}{R_0^5}\right)^{1/2}}{\frac{1}{2}\left(\frac{h_1 E_1}{R_1^5}\right)^{1/2} + \left(\frac{h_0 E_0}{R_0^5}\right)^{1/2}}
\end{aligned}
$$

If $E_1 = E_0$ *and* $\dfrac{h_1}{R_1} = \dfrac{h_0}{R_0}$,

$$
\frac{P_r}{P_i} = \frac{\frac{1}{2}\frac{1}{R_1^2} - \frac{1}{R_0^2}}{\frac{1}{2}\frac{1}{R_1^2} + \frac{1}{R_0^2}} = \frac{\frac{1}{2}\left(\frac{R_0}{R_1}\right)^2 - 1}{\frac{1}{2}\left(\frac{R_0}{R_1}\right)^2 + 1}
$$

For $2A_1 = A_0$, $R_1 = \dfrac{1}{\sqrt{2}} R_0$. $\left(\dfrac{R_0}{R_1}\right)^2 = 2$

$$
\frac{P_r}{P_i} = 0
$$

B. Suppose the vessels distal to the bifurcation are severely calcified, so that $E_1 \gg E_0$. What will the reflection coeffecient be? Does this suggest a noninvasive method of detecting the presence of severe arterial disease?

$Z_L \gg Z_0$, *so* $\dfrac{P_r}{P_i} > 1$ *and reflection is positive for positive wave in.*

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Problem 2

This problem deals with the estimation of the pressure drop to be expected across a vascular stenosis. Figure 2 is a sketch of the cross-section of a stenotic artery with a concentric plaque. An idealized model is shown in Figure 3, demonstrating a narrowed region followed by a sudden expansion where the fluid will generally exhibit turbulent flow. Energy will be lost in two ways: in viscous flow in the narrow region and in turbulent loss in the expansion. In this problem we will concentrate on the latter.

Figure 2:

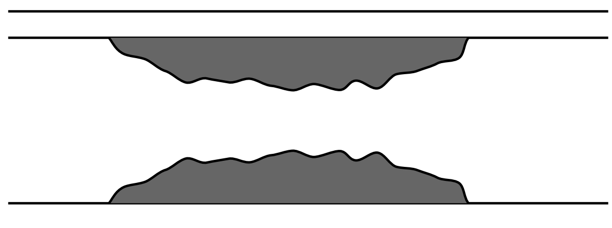
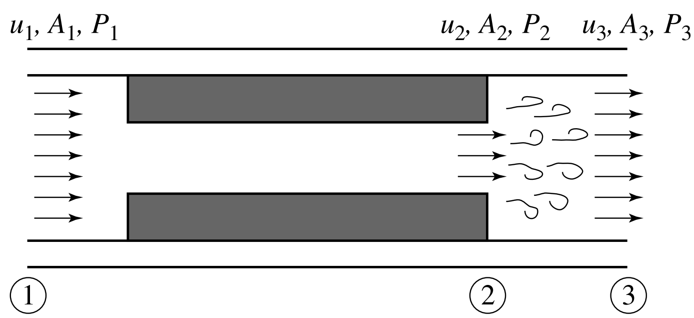


Figure 3:



The first task will be to estimate the pressure drop from point 2 to point 3. Position 3 is chosen far enough downstream to be in a region of uniform flow, of velocity u_3. In order to solve this problem we must make use of the "linear momentum theorem".

Consider a flow field of fluid with a superimposed control volume (CV). Newton's second law for the system included within the CV is

$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = \sum \vec{F}_{system} \tag{1}$$

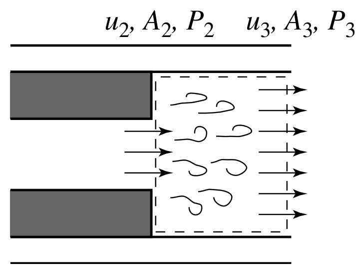
This equation states that the rate of change of the system momentum equals the sum of forces acting on the mass inside the control volume. Equation 1 may be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_{\mathrm{CV}}} \rho\vec{U}\,\mathrm{d}V + \oint_{A_{\mathrm{CV}}} \rho\vec{U}\left(\vec{U}\cdot\vec{n}\right)\mathrm{d}A = \sum \vec{F} \tag{2}$$

The first term is the rate of accumulation of momentum within the control volume; the second term is the net rate of momentum flux *out* of the CV.

Let us establish a CV for our model of the sudden expansion in Figure 3. (The volume within the dashed line in Figure 4.)

Figure 4:



The fluid is incompressible, and there is no net acceleration of the mass within the CV, so the first term of Equation 2 disappears. How do we evaluate the second term?

A. Calculate the momentum flux *into* the CV. (Note that flux is the rate of momentum passing into the CV per unit time.) Do you agree that the answer is:

$$\rho u_2(u_2 A_2) = \rho A_2 u_2^2$$

Momentum flux into *the CV is*

$$\underbrace{\rho u_2}_{\text{momentum per unit volume}} \times \underbrace{U_2 a_2}_{\text{volume flow per unit time}} = \rho A_2 U_2^2 \tag{3}$$

B. What is the momentum flux *out* of the CV at point 3?

Momentum flux out of *the CV is*

$$\rho A_3 u_3^2 \tag{4}$$

Thus, the net *momentum flux out of the CV is*

$$\rho A_3 u_3^2 - \rho A_2 u_2^2 \tag{5}$$

C. Use conservation of mass to relate u_3 to u_2 and the cross-sectional areas A_2 and A_3.

Conservation of mass states that

$$\begin{aligned} u_2 A_2 &= u_3 A_3 \\ u_3 &= u_2 \frac{A_2}{A_3} \end{aligned} \tag{6}$$

D. Next we need to calculate the right hand side of Equation 2. The forces acting on the CV are *pressure* forces. Assume that the pressure at the entrance orifice, P_2, is equal across the entire face of the CV at position 2. What is the *net* force acting on the CV?

Net force *from pressure would be:*

$$P_2 A_3 - P_3 A_3 = A_3(P_2 - P_3) \tag{7}$$

E. What is the pressure drop $P_2 - P_3$ as a function of the velocity u_2, the areas A_2 and A_3, and the fluid density?

We may rewrite Equation 5 as

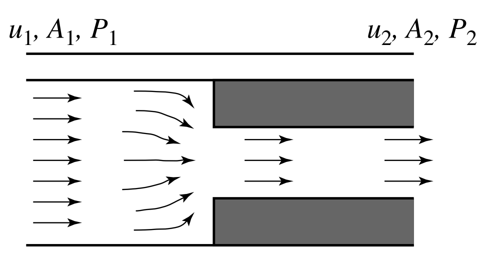
$$\rho A_3 u_2 \frac{A_2^2}{A_3^2} - \rho A_2 u_2^2 = \rho u_2^2 A_2 \left(\frac{A_2}{A_3} - 1 \right) \tag{8}$$

From (8), (7) we have

$$\begin{aligned} \sum \vec{F} &= \oint \rho \vec{u}(\vec{u} \cdot \vec{n}) \mathrm{d}A \\ (P_2 - P_3)A_3 &= \rho u_2^2 A_2 \left(\frac{A_2}{A_3} - 1 \right) \\ P_2 - P_3 &= \rho u_2^2 \frac{A_2}{A_3} \left(\frac{A_2}{A_3} - 1 \right) \end{aligned} \tag{9}$$

F. Now let us focus on the entrance portion of the stenosis shown below, and consider inviscid flow (Figure 5).

Figure 5:



Use the Bernoulli principle and conservation of mass to relate the pressure drop $P_1 - P_2$ to the characteristics of the fluid, the entrance velocity (u_1), and the geometry.

Apply Bernoulli principle between (3) and (4):

$$\begin{aligned}
P_1 + \frac{1}{2}\rho u_1^2 &= P_2 + \frac{1}{2}\rho u_2^2 \\
u_2 &= u_1 \frac{A_1}{A_2} \qquad \textit{(continuity)} \\
P_1 - P_2 &= \frac{1}{2}\rho\left(u_2^2 - u_1^2\right) \\
P_1 - P_2 &= \frac{1}{2}\rho u_1^2 \left[\left(\frac{A_1}{A_2}\right)^2 - 1\right] \qquad (10)
\end{aligned}$$

G. If we neglect viscous losses in the region of the stenosis, what will be the total pressure drop $P_1 - P_3$? Your derivation should end up with the following:

$$P_1 - P_3 = \frac{1}{2}\rho u_1^2 \left[\frac{A_1}{A_2} - 1\right]^2$$

$$P_1 - P_3 = (P_1 - P_2) + (P_2 - P_3)$$

From (9) and (10) we have

$$P_1 - P_3 = \rho u_2^2 \frac{A_2}{A_3}\left(\frac{A_2}{A_3} - 1\right) + \frac{1}{2}\rho u_1^2 \left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

But $u_2 = u_1 \frac{A_1}{A_2}$*, and* $A_1 = A_3$*; thus*

$$\begin{aligned}
P_1 - P_3 &= \frac{1}{2}\rho u_1^2 \left[\frac{A_1}{A_2}\left(\frac{A_2}{A_1} - 1\right) + \left(\frac{A_1}{A_2}\right)^2 - 1\right] \\
P_1 - P_3 &= \frac{1}{2}\rho u_1^2 \left(\frac{A_1}{A_2} - 1\right)^2 \qquad (11)
\end{aligned}$$

H. In a vessel where u_1 is 30 cm/sec, and the ratio A_2/A_1 is 0.1, estimate the pressure drop in mmHg. (Remember that 1 mmHg=1330 dynes/cm^2.) What pressure drop would be expected if the lumen *diameter* is reduced to 25% of its original size?

$$\begin{aligned}
P_1 - P_3(mmHg) &= \frac{1}{1330} \times \frac{1}{2} \times 1 \times 900 \frac{cm^2}{sec^2} \times (10-1)^2 \\
&= \frac{(900)(81)}{(1330)(2)} = \boxed{27.4\ mmHg}
\end{aligned}$$

If the diameter were reduced to one-fourth its original size, and area A_2 would be one-sixteenth the size of A_1. Thus

$$\frac{A_2}{A_1} = \frac{1}{16} \quad (6.25\%)$$

Using the same formula:

$$\begin{aligned} P_1 - P_3 &= \frac{1}{1330} \times \frac{1}{2} \times 1 \times 900 \times (16-1)^2 \\ &= \frac{(900)(225)}{(1330)(2)} = \boxed{76\ mmHg} \end{aligned}$$

Note therefore the rapid increase in pressure drop across a stenosis as the area ratio drops below 10%. A drop in diameter by a factor of four certainly warrants the term "critical" stenosis.

Realize, of course, that in real life as the stenosis increases, the flow velocity u_1 will in general not be maintained, but will drop *as blood is shunted elsewhere.*

$$P_1 - P_3(mmHg) = \underbrace{\frac{1}{1330} \times \frac{1}{2} \times 1 \times 30^2}_{0.34} \times \left(\frac{A_1}{A_2} - 1\right)^2$$

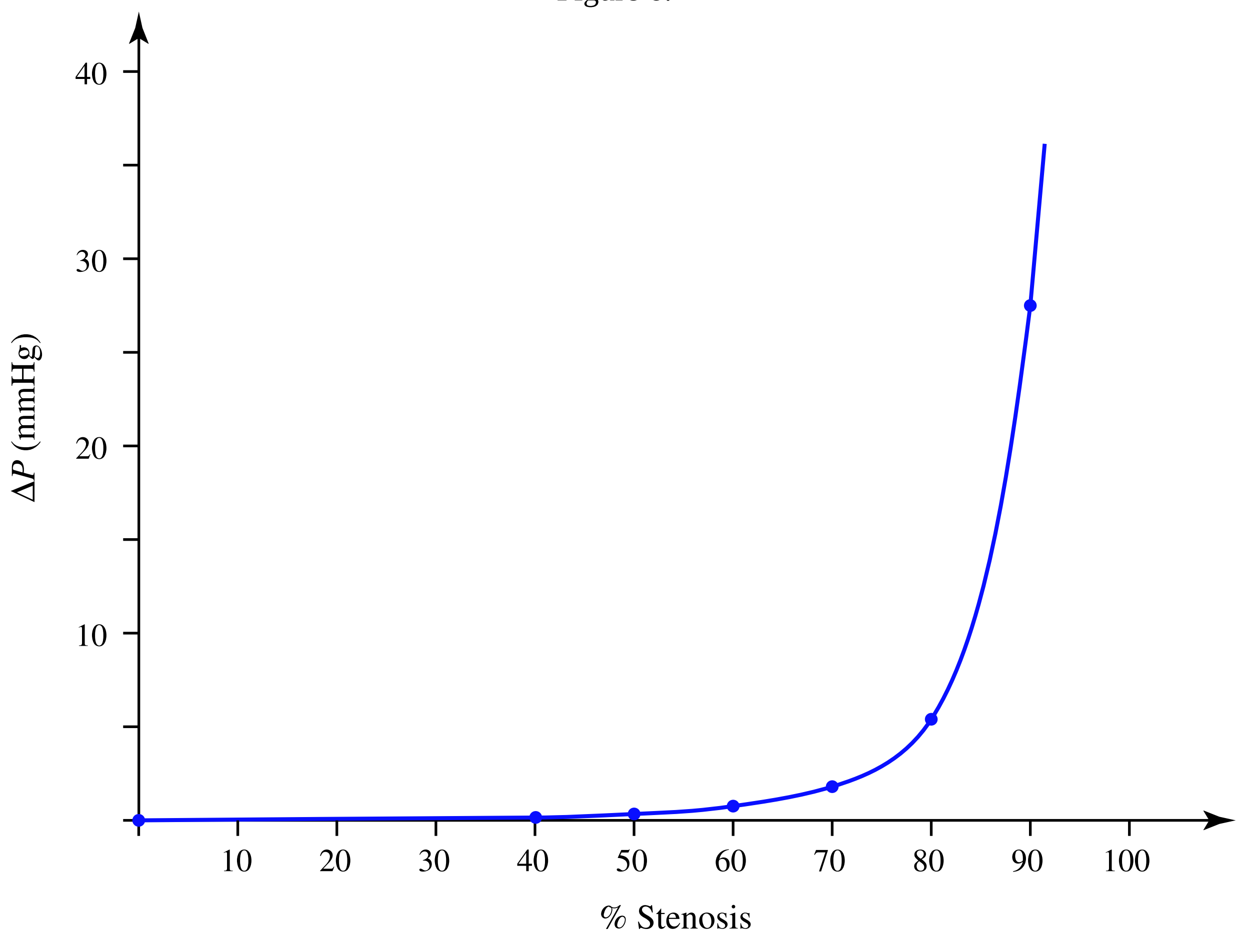
If $S \equiv \%$ stenosis $= \left(\frac{1-A_2}{A_1}\right) \times 100$

Then

$$\left[\frac{A_1}{A_2} - 1\right] = \frac{S}{100 - S}$$

(See Figure 6.)

Figure 6:



S (%)	$\frac{S}{100-S}$	$\left[\frac{S}{100-S}\right]^2$	ΔP (mmHg) $(0.34)\left(\frac{S}{100-S}\right)^2$
0	0	0	0
10	.11	.012	.004
40	.67	.45	.15
50	1.0	1.0	.34
60	1.5	2.25	.76
70	2.3	5.29	1.8
80	4	16	5.4
90	9	81	27.5

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Problem 3

Engineers for a medical products company have developed an inexpensive, hand-held device for monitoring peak expiratory flow rates in asthmatics. The device, shown in Figure 7, consists of a rigid cylinder with a long, narrow slit down the side and a close-fitting, spring-loaded piston diaphragm. Expired air enters the chamber of the peak flow meter from the left through the mouth-piece, then exits via the slit *at a uniform velocity.* The length of the slit available for flow, x, is equal to the displacement of the piston. Gas exits from the slit in the form of a *jet* into the atmosphere.

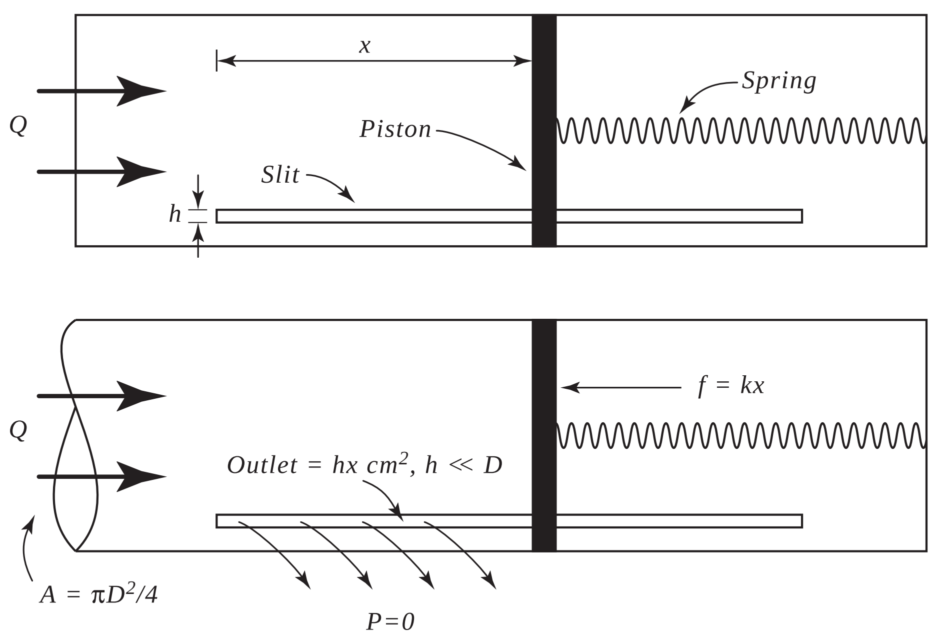
Assuming the flow to be *quasi-steady and inviscid*, that the piston moves without friction, and that the displacement is linearly proportional to the force F acting on it:

$$F = kx$$

answer the following questions. You may take the following as given:

- density of expired air, ρ
- piston diameter, D
- slit height, h ($h \ll D$)
- volume flow rate, Q
- spring constant, k
- exposed slit length, x

Figure 7:



Note: The x-direction linear momentum equation cannot *be used in this problem because the problem is under-specified. In particular, the restraining force on the cylinder to hold it against the mouth is not given.*

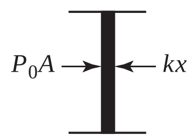
A. Given the volume flow rate entering the device, Q, and assuming the piston to be displaced a distance x but *non-accelerating*, what is the velocity of gas exiting through the slit?

Conservation of mass:

$$\begin{aligned} \text{area of slit} &= hx \\ Q &= v_{slit} \cdot hx \\ v_{slit} &= \frac{Q}{hx} \end{aligned}$$

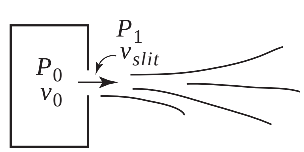
B. What is the pressure inside the chamber of the flowmeter, p_o?

Because we're told that the piston is not accelerating, we know that the forces on the piston must cancel.



$$\begin{aligned} P_0 A &= kx \\ A &= \frac{\pi D^2}{4} \\ P_0 &= \frac{kx}{A} = \frac{4kx}{\pi D^2} \end{aligned}$$

Can also use Bernoulli across slit:



$$\begin{aligned} P_0 + \frac{1}{2}\rho v_0^2 &= P_1 + \frac{1}{2}\rho v_{slit}^2 \\ \text{But } P_1 &= 0 \text{ and assume } v_{slit} \gg v_0 \\ P_0 &= \frac{1}{2}\rho v_{slit}^2 = \frac{1}{2}\rho \frac{Q^2}{h^2 x^2} \end{aligned}$$

C. What is the relationship between the displacement of the piston, x, and the other given parameters?

Use Bernoulli across slit:

$$\begin{aligned}
P_0 + \frac{1}{2}\rho v_0^2 &= P_{out} + \frac{1}{2}\rho v_{slit}^2 \\
\textit{But } P_{out} &= 0 \qquad \textit{and assume } v_{slit} \gg v_0 \\
P_0 &= \frac{1}{2}\rho v_{slit}^2 = \frac{1}{2}\rho\frac{Q^2}{h^2x^2} = \frac{4kx}{\pi D^2} \qquad \textit{(from B)} \\
\textit{So} & \\
x^3 &= \frac{\pi}{8}\frac{\rho D^2 Q^2}{kh^2} \\
x &= \left[\frac{\pi}{8}\frac{\rho D^2 Q^2}{kh^2}\right]^{1/3}
\end{aligned}$$

If Bernoulli was used in part B, then here we must balance the forces:

$$\begin{aligned}
P_0 A &= kx \\
\frac{1}{2}\rho\frac{Q^2}{h^2x^2} \cdot \frac{\pi D^2}{4} &= kx \\
x^3 &= \frac{\pi}{8}\frac{\rho D^2 Q^2}{kh^2} \\
x &= \left[\frac{\pi}{8}\frac{\rho D^2 Q^2}{kh^2}\right]^{1/3}
\end{aligned}$$

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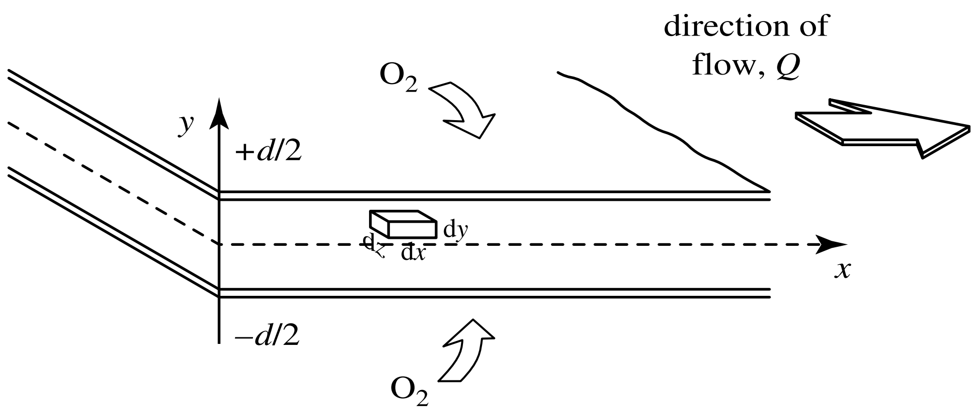
Problem 4

You are conducting experiments in your laboratory on a new artificial lung which consists of two parallel membranes separated by a small distance d. (See Figure 8.) Blood flows through the gap under steady laminar conditions due to a pressure gradient, $-\mathrm{d}p/\mathrm{d}x$, and oxygen transport occurs across the membranes. Assume blood to be a Newtonian fluid of viscosity μ.

Figure 8:



A. Using the coordinate system shown in Figure 8, what are the *boundary conditions* on the fluid velocity $v_x(y)$ and the shear rate $\dot{\gamma} = \dfrac{\partial v_x(y)}{\partial y}$?

1: No-slip condition: $V_x = 0$ *at* $y = \pm d/2$

2: Shear rate zero at axis of symmetry: $\mathrm{d}V_x/\mathrm{d}y = 0$ *at* $y = 0$

B. Set up the differential equation of motion for the control volume $\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$.

For steady laminar flow, the rate of change of momentum is zero, hence the sum of all forces acting on the control volume must be zero.

$$\begin{aligned}
\textit{Force due to pressure gradient: } F_p &= -\frac{\mathrm{d}}{\mathrm{d}x}(P(x)\mathrm{d}y\mathrm{d}z)\mathrm{d}x = \frac{-\mathrm{d}P(x)}{\mathrm{d}x}\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
\textit{Force due to viscous shear: } F_v &= \frac{\mathrm{d}}{\mathrm{d}y}\left[\tau(y)\mathrm{d}x\mathrm{d}z\right]\mathrm{d}y = \frac{-\mathrm{d}\tau(y)}{\mathrm{d}y}\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
F_p + F_v &= 0 \\
\frac{-\mathrm{d}\tau(y)}{-}\frac{-\mathrm{d}P(x)}{\mathrm{d}x} &= 0 \quad \textit{where } \frac{-\mathrm{d}P(x)}{\mathrm{d}x} = \textit{constant and } \tau(y) = \mu\frac{\mathrm{d}V_x(y)}{\mathrm{d}y} \\
\textit{or} \quad \mu\frac{\mathrm{d}^2V_x(y)}{\mathrm{d}y^2} &= \frac{\mathrm{d}P(x)}{\mathrm{d}x}
\end{aligned}$$

C. Solve the equation (subject to the boundary conditions) for the blood velocity in the x-direction as a function of y. Sketch the result.

Solving the differential equation by integrating once:

$$\frac{dV_x}{dy} = \frac{1}{\mu} \cdot \frac{dP}{dx} \cdot y + c_1$$

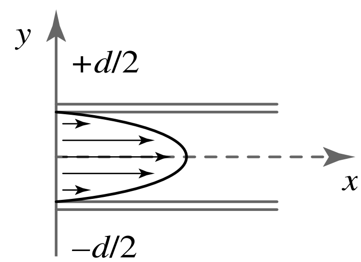
Boundary condition 2 requires $c_1 = 0$.

Integrating again:

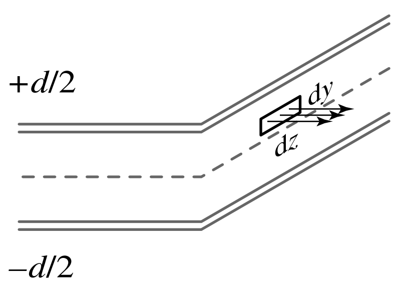
$$V_x(y) = \frac{1}{2\mu} \cdot \frac{dP}{dx} \cdot y^2 + c_2$$

Using boundary condition 1

$$V_x(y) = \frac{1}{2\mu}\left(\frac{-dP}{dx}\right) \cdot \left[\left(\frac{d}{2}\right)^2 - y^2\right]$$



D. Using the result of (C), determine the flow rate per unit width, Q/dz, in terms of the pressure gradient $-dp/dx$, the gap width d, and the blood viscosity μ.



$$Q = 2\int_0^{d/2} V_x(y)\,dy\,dz$$

$$\frac{Q}{dz} = 2\int_0^{d/2} \frac{1}{2\mu}\left(-\frac{dP}{dx}\right)\left[\left(\frac{d}{2}\right)^2 - y^2\right] dy$$

$$\frac{Q}{dz} = \frac{d^3}{12\mu}\left(-\frac{dP}{dx}\right)$$

E. What is the resistance, R, per unit width of the structure?

$$\textit{Resistance} = \frac{\Delta P}{\frac{Q}{dz}} = \frac{12\mu\Delta x}{d^3} = \frac{12\mu L}{d^3} \textit{ where } \Delta x = L$$

F. What is the shear rate at the membrane surface in terms of the average flow velocity, $\bar{V}$?

The shear rate at the membrane surface is given by

$$\begin{aligned} \dot{\gamma}|_{wall} &= \left.\frac{\partial V_x}{\partial y}\right|_{y=d/2} \\ &= \frac{1}{\mu}\frac{\partial P}{\partial x}\frac{d}{2} \end{aligned}$$

We can obtain the average velocity from total flow and cross-sectional area:

$$\bar{V} = \frac{Q}{A} = \frac{Q}{d} = \frac{d^2}{12\mu}\left(-\frac{\mathrm{d}P}{\mathrm{d}x}\right)$$

Thus

$$\dot{\gamma} = \bar{V} \cdot \frac{6}{d}$$

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