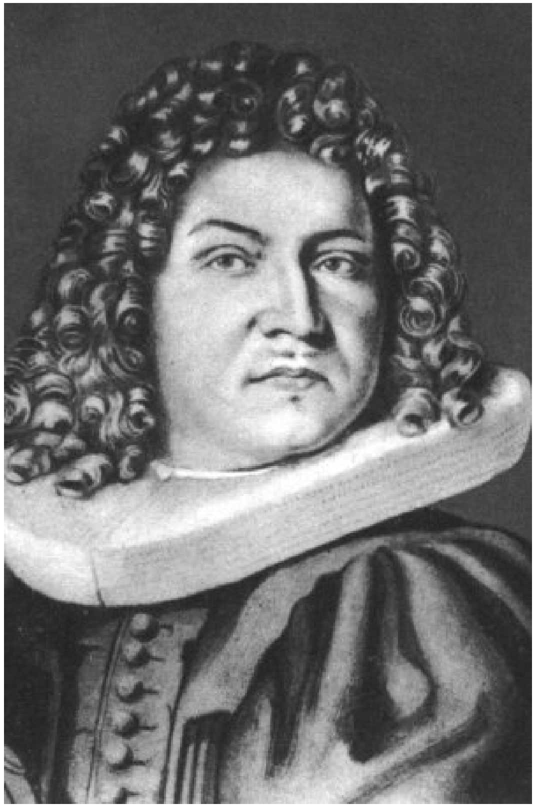

Source: Wikipedia

Jacob (James) Bernoulli
(1654–1705)
For Bernoulli Trials

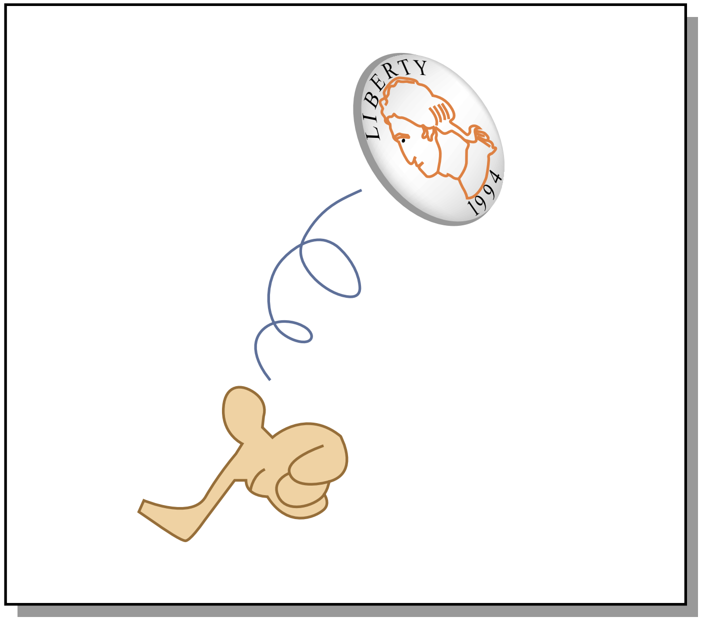

Figure by MIT OCW.

ESD.86
Class #2

February 12, 2007

Analyzing a Probability Problem
Four Steps to Happiness

1. Define the Random Variable(s)
2. Identify the (joint) sample space
3. Determine the probability law over the sample space
4. Carefully work in the sample space to answer any question of interest

Subway Interviews

You are standing outside of the Park Street subway (T) station, with clipboard, and you want to interview only registered Republicans. For modeling purposes we will say that the probability that a random T-rider who passes you is a Republican is 0.20. (Actual is 0.13.) You have infinite patience, as the stream of riders continues all day long.

(a) Let R = the number of T-riders you question until you find the 1st Republican. Find the probability mass function of R.

Geometric Probability Mass Function

$$P\{R = n\} = (0.8)^{n-1}(0.2) \quad n = 1,2,3,\ldots$$

$$E[R] = \sum_{n=1}^{\infty} nP\{R = n\} = \sum_{n=1}^{\infty} n(0.8)^{n-1}(0.2) =$$

$$E[R] = \left[\frac{d}{dz}\sum_{n=1}^{\infty} z^{n}(0.8)^{n-1}(0.2)\right]_{z=1} = \left[\frac{d}{dz}(0.2z)/(1 - z0.8)\right]_{z=1}$$

$$E[R] = \frac{0.2(1 - 0.8) - 0.2(-0.8)}{(1 - 0.8)^{2}} = \frac{0.2}{(0.2)^{2}} = 1/0.2 = 5$$

Here we used the z-transform for the probability mass function (PMF), defined as

$$p_X^T(z) \equiv \sum_{x=0}^{\infty} p_X(x) z^x, \quad |z| \leq 1, \text{ where}$$

$$p_X(x) \equiv P\{X = x\}, x = 0,1,2,\ldots$$

$$p_X^T(z) \equiv \sum_{x=0}^{\infty} p_X(x) z^x, \quad |z| \leq 1$$

$$\frac{d}{dz} p_X^T(z) = \sum_{x=0}^{\infty} x p_X(x) z^{x-1}$$

$$\left.\frac{d}{dz} p_X^T(z)\right]_{z=1} = \sum_{x=0}^{\infty} x p_X(x) = E[X]$$

Both discrete and continuous (Laplace) transforms will be reviewed in tutorial on Friday.

If N people go by, how many did you interview?

X=number interviewed, X=0,1,2,…

p=0.2.

$$P\{X=j\}=\binom{N}{j}p^j(1-p)^{N-j}, j=0,1,2,\ldots,N$$

Binomial Probability Mass Function

How many people go by you until you have completed your k^{th} interview?

Let Y=number of people who pass by up to and including the one who is the k^{th} person interviewed.

Negative Binomial Probability Mass Function

$$P\{Y=y\}=\binom{y-1}{k-1}p^{k}(1-p)^{y-k}, k=0,1,\ldots,k; y=1,2,\ldots$$

Indicator random variables.

Suppose $X_i = \begin{cases} 1 & \text{with probability } p_{\mathrm{i}} \\ 0 & \text{with probability } 1 - p_{\mathrm{i}} \end{cases}$

Then $E[X_i] = 1*p_i + 0*(1- p_i) = p_i$.

Example 1: Flip a coin N times, with P{Heads} = p. Assume independent flips.

Define the random variable NH = number of Heads in N flips.

Let the set indicator R.V. X_i be 1 if the i^{th} coin flip is Heads.

$$E[NH] = E[\sum_{i=1}^{N} X_i] = \sum_{i=1}^{N} E[X_i] = \sum_{i=1}^{N} p = Np$$

Set indicator random variables.

Suppose $$X_i = \begin{cases} 1 & \text{with probability } p_\text{i} \\ 0 & \text{with probability } 1 - p_\text{i} \end{cases}$$

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Example 2. Baseball Hats.

Suppose that one player from each of the 30 Major League Baseball teams attends a party at MIT, and each arrives wearing his team's baseball cap. Each tosses his hat into a closet upon arrival. The host, at the end of the party, gives a ***random hat*** to each departing player. What is the expected number of hats that are returned to their rightful owners?

Solution: Define indicator r.v. $X_i = \begin{cases} 1 \text{ if player } i \text{ gets his hat back} \\ 0 \text{ otherwise} \end{cases}$

Let H = the number of hats returned to the correct owners. Then,

$$E[H] = E[\sum_{i=1}^{30} X_i] = \sum_{i=1}^{30} E[X_i] = \sum_{i=1}^{30} (1/30) = 1.$$

Answer independent of the number of players or teams!

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Answer independent of the number of players or teams!

Example #3. Winning Streaks.

Suppose the Boston Celtics were much better, that they had a 50% chance of winning any game, and that the outcomes of all games were mutually independent.

If there were 100 games in a season, how many 7 game winning streaks might we expect?

How do we interpret this?

(1) Game *n* is a game which they win and for which they have won the previous 6. Here $(1/2)^7 = 1/128$. Use indicator r.v.'s.

(2) Game *n* is a game which they win and for which they have won the previous 6, *and* they lose the next game. Here we need $(1/2)^7\ (1/2) = 1/256$.

Great Expectations.

Suppose you flip a coin until you get the first Heads, then stop. If the 1^{st} flip is Heads, you win \$2. If the 2nd is the 1^{st} Heads, you win \$4.

If the 3rd flip is the 1^{st} Heads, you win \$8. …If the n^{th} flip is the 1^{st} Heads, you win $\$2^n$. The bank is Donald Trump, so this game can go on for a long long time!

(a) How much would you be willing to pay for playing this game?

(b) What is the expected dollar value of winnings in the game?

$$E[D] = 2(1/2) + 4(1/2)^2 + 8(1/2)^3 + \ldots + 2^n (1/2)^n + \ldots$$

$$E[D] == 1 + 1 + 1 + 1 + 1 + \ldots = \infty$$

This is the *St.Petersburg Paradox*

Daniel Bernoulli (1738; English trans. 1954)

‘Google’ this & find many interesting articles, such as

http://plato.stanford.edu/entries/paradox-stpetersburg/

Does the St. Petersburg Paradox Occur in Nature?

A possible term project!

Size and frequency of occurrence of Earthquakes.

Small earthquakes occur every day all around the world, ….Large earthquakes occur less frequently, the relationship being exponential; namely, roughly ten times as many earthquakes larger than magnitude 4 occur in a particular time period than earthquakes larger than magnitude 5. In the (low seismicity) United Kingdom, for example, it has been calculated that the average recurrences are:•an earthquake of 3.7 or larger every Year, an earthquake of 4.7 or larger every 10 years, an earthquake of 5.6 or larger every 100 years. …

The USGS estimates that, since 1900, there have been an average of 18 major earthquakes (magnitude 7.0-7.9) and one great earthquake (magnitude 8.0 or greater) per year, and that this average has been relatively stable.[5

http://en.wikipedia.org/wiki/Earthquake#Size_and_frequency_of_occurrence

Earthquakes

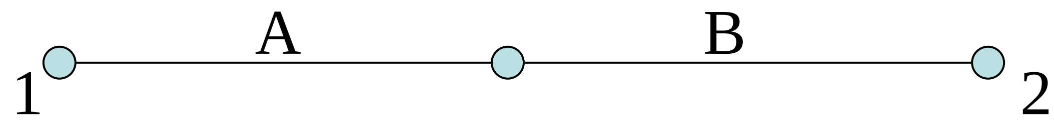
- Richter Scale: logarithmic. Each whole number step in the magnitude scale corresponds to the release of about 31 times more energy than the amount associated with the preceding whole number value.

		Richter Magnitude
Seismic Energy Yield	**Example**	0.5
5.6 kg (12.4 lb)	Hand grenade	1.0
32 kg (70 lb)	Construction site blast	1.5
178 kg (392 lb)	WWII conventional bombs	2.0
1 metric ton	late WWII conventional bombs	2.5
5.6 metric tons	WWII blockbuster bomb	3.0
32 metric tons	Massive Ordnance Air Blast bomb	3.5
178 metric tons	Chernobyl nuclear disaster, 1986	4.0
1 kiloton	Small atomic bomb	4.5
5.6 kilotons	Average tornado (total energy)	5.0
32 kiloton	Nagasaki atomic bomb	5.5
178 kilotons	Little Skull Mtn., NV Quake, 1992	6.0
1 megaton	Double Spring Flat, NV Quake, 1994	6.5
5.6 megatons	Northridge quake, 1994	7.0
50 megatons	Tsar Bomba, largest thermonuclear weapon ever tested	7.5
178 megatons	Landers, CA Quake, 1992	8.0
1 gigaton	San Francisco, CA Quake, 1906	8.5
5.6 gigatons	Anchorage, AK Quake, 1964	9.0
32 gigatons	2004 Indian Ocean earthquake	10.0

Does this fit the idea of the St. Petersburg Paradox?

http://www.answers.com/topic/richter-magnitude-scale

"…ilities": Reliability, Robustness



Let T_{12} = event successful Transmission from node 1 to node 2.

p_A=P{Link A works properly}=P{A}

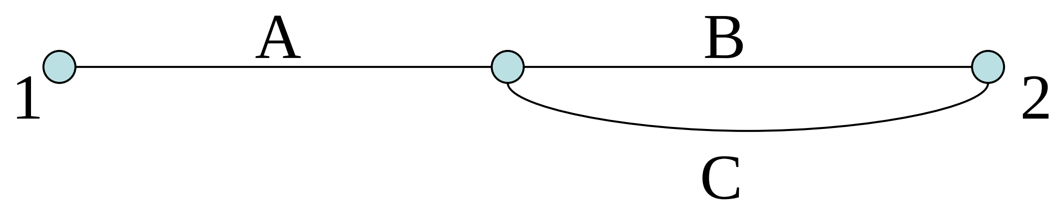
p_B= P{Link B works properly}=P{B}

Assume links A and B function independently.

Then,

$$\mathbf{P\{T_{12}\}=P\{A\ and\ B\}=P\{AB\}=} p_A p_B$$

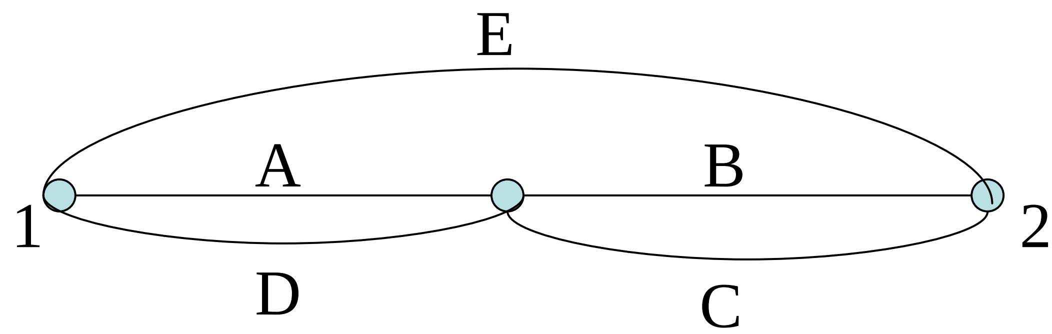
Add one redundant arc to increase Reliability



Let T_{12} = event successful Transmission from node 1 to node 2.
p_A=P{Link A works properly}=P{A}
p_B= P{Link B works properly}=P{B}
p_C= P{Link C works properly}=P{C}
Assume links A, B and C function independently.
Then,
P{T_{12}}=P{A and (B or C)}=P{AB+AC}= P{AB+ACB'}=

$$\mathbf{P\{T_{12}\} = p_A p_B + p_A p_C (1 - p_B)}$$

Add two more redundant arcs to increase Reliability



Let T_{12} = event successful Transmission from node 1 to node 2.
Assume all links function independently.
Then,

$P\{T_{12}\}=P\{E \text{ or } (AB) \text{ or } (AC) \text{ or } (DB) \text{ or } (DC)\}$

$P\{T_{12}\}=P\{E + E'[(AB) + (AC) + (DB) + (DC)]\}$

$P\{T_{12}\}=P\{E + E'[A(B+C) + D(B +C)]\}$

$P\{T_{12}\}=P\{E + E'[(A+D)(B+C)]\}= P\{E + E'[(A+A'D)(B+B'C)]\}$

$$\mathbf{P\{T_{12}\}= p_E + (1-p_E)\{[p_A+(1- p_A)\, p_D]\, [p_B+(1- p_B)\, p_C]\}}$$

Generalization of this could be a term project.