<< Next time: problem ... >>

6.896
3/8/04
L9.1

Optimal retiming

Recall: $W(u,v) = \min\{w(p) : u \overset{p}{\leadsto} v\}$
$D(u,v) = \max\{d(p) : u \overset{p}{\leadsto} v$ is a critical path$\}$

Lemma 1. $\Phi(G) \le c$ iff $W(u,v) \ge 1$ whenever $D(u,v) > c$. $\boxtimes$

Lemma 2. Let r be legal retiming of G.

1. p is a crit path of G iff p is a crit path of G_r.
2. $W_r(u,v) = W(u,v) - r(u) + r(v)$
3. $D_r(u,v) = D(u,v)$. $\boxtimes$

Lemma 3. $\Phi(G_r) = D(u,v)$ for some $u, v \in V$.

Proof. Let $u \overset{p}{\leadsto} v$ be path in G_r \$ $\Phi(G_r) = d(p)$ and $w_r(p) = 0$ (def of clock period). Thus, $W_r(u,v) = w_r(p) = 0$, and $D_r(u,v) = d(p)$, since no 0-wt path in G_r has larger delay than p. Thus, $\Phi(G_r) = D_r(u,v) = D(u,v)$. $\boxtimes$

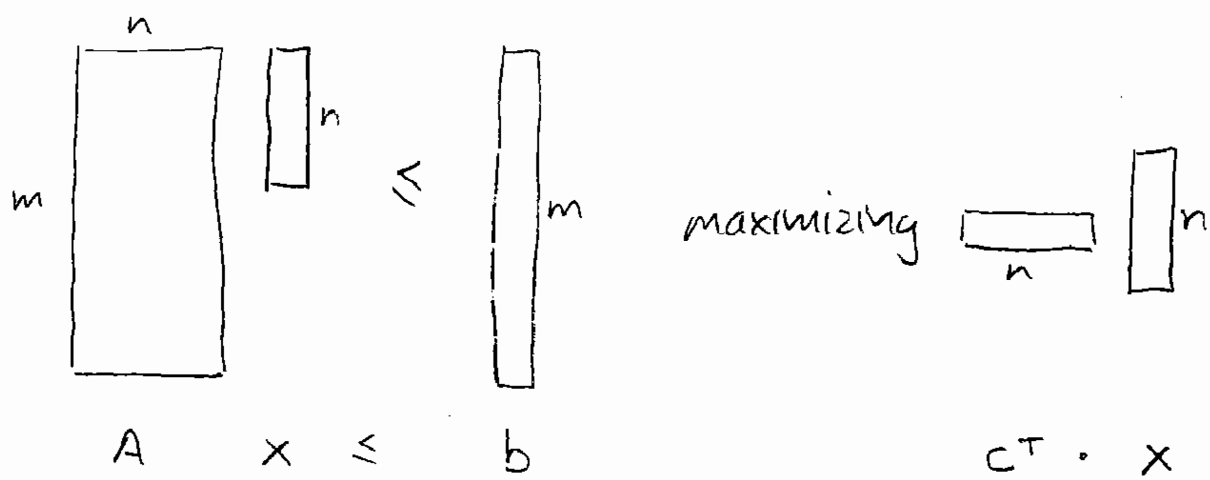
Theorem. Let $r: V \to \mathbb{Z}$. Then, r is a legal retiming of G \$ $\Phi(G_r) \le c$ iff
1. $r(u) - r(v) \le w(e)$ $\forall u \overset{e}{\to} v$.
2. $r(u) - r(v) \le W(u,v) - 1$ $\forall u, v$ \$ $D(u,v) > c$.

Proof.
1. $r(u) - r(v) \le w(e)$ iff $w_r(e) = w(e) - r(u) + r(v) \ge 0$
2. $\Phi(G_r) \le c$ precisely when $w_r(u,v) \ge 1$ $\forall u, v \in V$ \$ $D_r(u,v) > c$, by Lemma 1. Rewrite using Lemma 2. $\boxtimes$

$O(E)$ constraints of type 1. } Linear!
$O(V^2)$ " " " 2. }

Linear programming

Let A be an $m \times n$ matrix, b be an m-vector, and c be an n-vector. Find an n-vector x that maximizes $c^T x$ subject to $Ax \le b$, or determine no solution exists.



General algs
- simplex - practical, but w-c exp. time
- interior-pt algs - polytime, becoming practical.

"Feasibility" problem: No opt. criterion.
Find $x \ni Ax \le b$.

Difference constraints

Each row of A contains exactly one 1 and one -1, and rest are 0's.

Ex. $\left.\begin{aligned} x_1 - x_2 &\le 3 \\ x_2 - x_3 &\le -2 \\ x_1 - x_3 &\le 2 \end{aligned}\right\} \; x_j - x_i \le a_{ij}$

Solution
$x_1 = 3$
$x_2 = 0$
$x_3 = 2$

Linear prog., but simpler.

Constraint graph

- vertex v_i for each unknown x_i
- edge $v_i \longrightarrow v_j$ with weight a_{ij} if $x_j - x_i \le a_{ij}$ is constraint.

Thm. If constraint graph has neg-wt cycle, then no solution. (Constraints unsatisfiable).

Pf. Sup. cycle is $v_1 \to v_2 \to \cdots \to v_k \to v_1$.

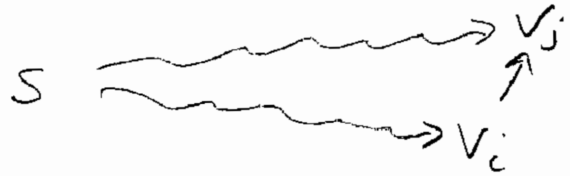
Then,
$$\begin{aligned}
x_2 - x_1 &\le a_{12} \\
x_3 - x_2 &\le a_{23} \\
&\vdots \\
x_k - x_{k-1} &\le a_{k-1,k} \\
x_1 - x_k &\le a_{k1} \\
\hline
0 &\le \text{wt of cycle} < 0
\end{aligned}$$

$\therefore$ No values for x_i satisfy constraints. $\boxtimes$

Thm. No neg-wt cycle $\Rightarrow$ constraints satisfiable.

Pf. Add new vertex s to V with 0-wt edge to each $v_i \in V$. (No neg-wt cycle introduced).

Let $\delta(s, v_i)$ = wt of sh. path from s to v_i.
(Sh. paths exist, since no neg-wt cycle)
Claim: $x_i = \delta(s, v_i)$ is solution.



$$\underset{\displaystyle x_j}{\underset{\shortparallel}{\delta(s,v_j)}} \le \underset{\displaystyle x_i}{\underset{\shortparallel}{\delta(s,v_i)}} + a_{ij} \quad (\Delta\text{-ineq.})$$

$\Rightarrow x_j - x_i \le a_{ij}$. $\boxtimes$

Bellman-Ford algorithm

Sh. path from source $s \in V$ to all $v \in V$ or determine neg-wt cycle exists.

Init: $d[v] = \begin{cases} 0 & \text{if } v = s \\ \infty & \text{otherwise.} \end{cases}$

for $i \leftarrow 1$ to $|V| - 1$
  do for each edge $(u, v) \in E$
    do $d[v] \leftarrow \min\{d[v], d(u) + w(u,v)\}$.

if $\exists\ (u,v)$ \$ $d[v] > d[u] + w(u,v)$
  then neg-wt cycle exists.

No neg-wt cycle $\Rightarrow$ $d[v] = \delta(s,v)$.

Correctness: induction (see CLRS)
Running time: $O(VE)$

Opt. clock period

1. Compute W and D — $O(V^3)$
2. Sort elems of D (clock period is one of them) — $O(V^2 \lg V)$
3. Binary search among D elems using Bellman-Ford to test feasibility of LP — $O(V^3 \lg V)$
4. Use values found by B-F to retime.

« Reminder: problem session next time »