Final Examination

- Do not open this exam booklet until you are directed to do so. Read all the instructions on this page.
- When the exam begins, write your name on every page of this booklet.
- This exam contains 5 problems, each with multiple parts. You have 80 minutes to earn 100 points.
- This exam booklet contains 9 pages, including this one. Two extra sheets of scratch paper are attached. Please detach them before turning in your exam at the end of the examination period.
- This exam is closed book. You may use two handwritten $8\frac{1}{2}'' \times 11''$ crib sheets. No calculators or programmable devices are permitted.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not waste time and paper rederiving facts that we have studied. It is sufficient to cite known results.
- Do not spend too much time on any one problem. Read them all through first, and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.
- Good luck!

Problem	*Parts*	*Points*	*Grade*	*Grader*
1	5	25		
2	4	16		
3	5	25		
4	3	18		
5	3	16		
Total	**20**	**100**		

Problem 1. Recurrences (5 parts) [25 points]

For each of the recurrences below, give an example of a problem during the term where this recurrence was important. Then, give an aysmptotically tight (Θ) bound for its solution.

(a) $f(n) = f(\lceil 2n/3 \rceil) + \Theta(1)$

Solution: Wallace tree: $\Theta(\lg n)$

(b) $f(n) = 2f(n/4) + \Theta(1)$

Solution: side of H-tree: $\Theta(\sqrt{n})$

(c) $f(n) = 2f(n/4) + \Theta(\sqrt{n})$

Solution: side of tree of meshes or area-universal fat-tree: $\Theta(\sqrt{n}\lg n)$

(d) $f(n) = f(n/2) + \Theta(\lg n)$

Solution: depth of bitonic sorting network: $\Theta(\lg^2 n)$

(e) $f(n) = 4f(n/4) + \Theta(\sqrt{n})$

Solution: depth of bitonic sorting network: $\Theta(n)$

Problem 2. Definitions (4 parts) [16 points]

Give a one-sentence description of each of the following terms to show that you know what they mean:

(a) carry-lookahead adder

Solution: Adding 2 N-bit numbers with $O(n)$ hardware, $O(\lg n)$ depth. The idea is to calculate the carries by doing parallel prefix in the "kpg" monoid.

(b) systolic circuit

Solution: a circuit formed of Moore machines. All state and communication with neighboring processors is clocked.

(c) load factor λ of a set M of messages on a network R.

Solution: $\lambda = \max_{\{\text{ all cuts of R }\}} \frac{\text{number of msgs crossing the cut}}{\text{capacity of the cut}}$

(d) Beneš network

Solution: Two butterflies back to back.

Problem 3. Bounds (5 parts) [25 points]

For each of the following problems, give the best possible bound. Since the upper and lower asymptotic bounds are the same for each answer, you can just give the bound without surrounding O's, Ω's, or Θ's. If you wish to justify your answer, please do so.

(a) The depth of a circuit for multiplying two n-bit numbers.

Solution: $\Theta(\lg n)$

(b) The depth of a circuit to rerepresent the sum of 3 n-bit binary numbers as the sum of 2 $(n+1)$-bit binary numbers.

Solution: $\Theta(1)$

(c) The time to sort n numbers on an $n \times n$ mesh-of-trees network.

Solution: $\Theta(\lg n)$

(d) The 3-dimensional volume to permute any n wires.

Solution: $\Theta(n^{3/2})$

(e) The time to simulate an n-node 3-dimensional mesh on an $\Theta(n \lg^2 n)$-area area-universal fat-tree with n leaves. (That is, simulate on a basic area-universal fat-tree, not the more complicated one with meshes at the leaves.)

Solution: $\Theta(n^{1/6})$ since $\lambda = \frac{n^{2/3}}{\sqrt{n}}$

Problem 4. Short Answer (3 parts) [18 points]

(a) Prove or give a counterexample. The number of latches in a synchronous circuit remains invariant under retiming.

Solution: This is false and counterexamples are easy to find. The correct statement is that the number of latches around a cycle is preserved.

(b) Give a Gray code on 4-bit numbers.

Solution:
0000
0001
0011
0010
0110
0111
0101
0100
1100
1101
1111
1110
1010
1011
1001
1000

(c) Consider an n-input, n-output butterfly network. Suppose that an adversary destroys a single switch in the network, preventing some inputs from connecting to some outputs in a single pass through the network. We can still find a large subset S of inputs and a large subset T of outputs such that every input in S can connect to every output in T. Let $m = \min\{|S|, |T|\}$, where S and T are chosen to allow m to be as large as possible. Give a good asymptotic upper bound on $n - m$.

Solution: $\Theta(\sqrt{n})$. Suppose a switch at level i is cut. There are 2^i inputs that can reach this switch and $2^{\lg n - i}$ outputs. Routing from any such input to any such output will be impossible, but all other paths still exist(the butterfly has unique paths between any input and output). So if we either exclude these inputs or these outputs from from the set of m, we will be able to route everything.

$$\text{max}_i \text{min}\{2^i, 2^{\lg n - i}\} = \sqrt{n}$$

Problem 5. Butterfly Fat-Tree (3 parts) [16 points]

Consider a butterfly fat-tree in which every 2 out of 3 levels consist of butterfly switches and every 1 out of 3 levels consists of tree switches.

(a) What is the asymptotic number of connections to the root of such an N-leaf fat-tree? Justify your answer.

Solution: Suppose the leaves are level 0. At each level i, let $C(i)$ be the number of connections going out of all nodes at level i. Then,

$$C(i) = C(i-1)$$

if i mod 3 is 0 or 1, and

$$C(i) = C(i-1)/2$$

if i mod 3 is 2, and hence

$$C(i) = C(i-3)/2 \forall i > 0, C(0) = n$$

There are $\lg n$ levels $\Rightarrow C(\lg n) = \Theta(\frac{n}{2^{\lg n/3}}) = \Theta(n^{2/3})$.

(b) How many switches does such an N-leaf fat-tree contain? Justify your answer.

Solution: Note that the number of switches in one node is Θ(number of out connections), so the answer is $\Theta(\sum_{i=0}^{\lg n} C(i)) = \sum \Theta(\frac{n}{2^{i/3}}) = \Theta(n)$

(c) Make an educated guess as to what important property this fat-tree might have.

Solution: It is $O(\lg V)$ universal for simulating volume V networks.

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