6.890	Lecture 23	Nov. 25, 2014

(guest lecture by Costis Daskalakis)

Recall:

- existence theorems: Nash, Brouwer, Sperner
- → total NP search problems
- Parity Arguments in Directed graphs (PAD)
- PPAD = class of Problems reducible to:
- End of the Line
  - circuits P & N: maps on n-bit node ids
  - edge $(v, w) \Leftrightarrow N(v) = w \wedge P(w) = v$
  - if given vertex is degree 1, find another
- Arithmetic Circuit SAT
  - circuit with cycles, no inputs, arbitrary fanout
  - total & PPAD-complete

Graphical games: [Kearns, Littman, Singh 2001]

- motivation: geographic/otherwise limited interaction
- players = nodes in a graph
- payoff depends only on your own & neighbors' strats.

Polymatrix games: [Janovskaya 1968]

- graphical games with edge-wise separable utility functions:

$$u_{w,v}(x_v, x_w) = \mathbb{E}_{\substack{s_u \sim x_u \\ s_w \sim x_w}} u_{w,v}(s_v, s_w) = \sum_{s_u, s_w} u_{w,v}(s_v, s_w) \cdot x_v(s_v) \cdot x_w(s_w)$$

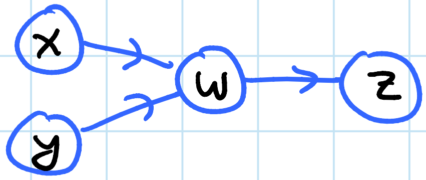
mixed strategies

PPAD-completeness of Nash: [Daskalakis, Goldberg, Papadimitriou 2006]

- reduction from Arithmetic Circuit SAT
- each player's strategy $\in \{0,1\}$

$\Rightarrow$ mixed strategy $\in [0,1]$

- addition gadget:



- w paid expected $\begin{cases} \$\Pr\{x:1\}+\Pr\{y:1\} & \text{if plays } 0 \\ \$\Pr\{z:1\} & \text{if plays } 1 \end{cases}$

  via matrix

$$\begin{array}{c|cc} \boxed{w:0} & y:0 & y:1 \\ \hline x:0 & 0 & 1 \\ x:1 & 1 & 2 \end{array} \qquad \begin{array}{cc} \boxed{w:1} & \\ z:0 & z:1 \\ \hline 0 & 1 \end{array}$$

- z is paid to play opposite of w:

$$u(z:0) = 0.5 \qquad u(z:1) = 1 - \Pr\{w:1\}$$

$\Rightarrow$ in any Nash equilibrium:

$$\Pr[z:1] = \min\{\Pr[x:1]+\Pr[y:1],\ 1\}$$

- $\Pr[z:1] < \min\{\Pr[x:1]+\Pr[y:1],\ 1\}$

  $\Rightarrow \Pr[w:0] = 1$

  $\Rightarrow \Pr[z:1] = 1$ $\times\!\!\times$ contradiction

- $\Pr[z:1] > \Pr[x:1]+\Pr[y:1]$

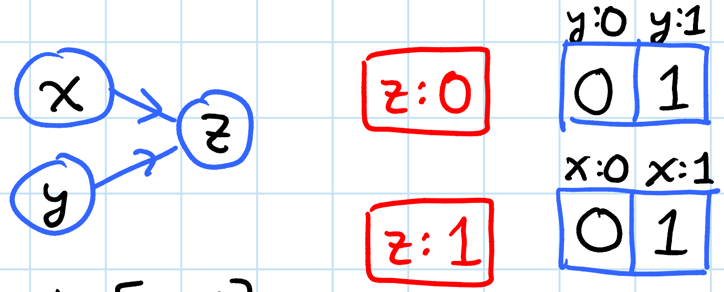
  $\Rightarrow \Pr[w:1] = 0$

  $\Rightarrow \Pr[z:1] = 0$ $\times\!\!\times$ contradiction

- subtraction gadget: just change:

$$\begin{array}{c|cc} & y:0 & y:1 \\ \hline x:0 & 0 & -1 \\ y:1 & 1 & 0 \end{array}$$

- comparison gadget:



  - $\Pr[x:1] > \Pr[y:1] \Rightarrow \Pr[z:1] = 1$
- etc. → polymatrix game

⇒ Arithmetic Circuit SAT is total (via Nash theorem) with rational. polynomial solution complexity solution

<u>Polymatrix game → 2-player game</u>: [Chen & Deng 2006] (almost in DGP 2006)
- make previous game bipartite
- one player ("lawyer") per color class
  - strategies = <u>union</u> of vertices' strategies
    ↳ pure ←
  - payoffs only along edges of directed graphs
- <u>hope</u>: Nash equilibrium of lawyer game
  ⇒ marginal distributions are Nash equilibrium of polymatrix game
- but lawyers prefer to represent lucrative clients
- <u>fix</u>: additional high-stakes game
  - if red lawyer plays strategy for red vertex i & blue lawyer plays strategy for blue vertex j then
    - if $i \neq j$ then both get \$0
    - if $i = j$ then red gets $+\infty$ ($2n \cdot u_{max}$) & blue gets $-\infty$
  ⇒ lawyers uniformly choose which vertex to rep.

- game = sum of two games
- in any Nash equilibrium:
  - close to uniform on vertices: $\forall u, v$
    $$x_u = \frac{1}{n}\left(1 \pm \frac{2 u_{max} n^2}{\infty}\right), \text{ ditto for } y_v$$
  - within vertex, payoff difference for red lawyer between $u{:}i$ & $u{:}j$ is $\sum_v \sum_\ell \left(A^{(u,v)}_{i,\ell} - A^{(u,v)}_{j,\ell}\right)$ (no ∞)
  - $\Rightarrow$ if $x_{u:i} > 0$ then $\forall j$: $\sum_u \sum_\ell \left(A^{(u,v)}_{i,\ell} - A^{(u,v)}_{j,\ell}\right) \geq 0$
  - define marginals $\hat{x}_u(i) = \frac{x_{u:i}}{x_u}$ & $\hat{y}_v(j) = \frac{y_{v:j}}{y_v}$
  - get approximate Nash equilibrium (would be exact if uniform on players)
  - set ∞ large
  - all these problems were hard to ε-approximate

<u>Easy 2-player Nash:</u> payoff matrices R & C
- zero sum $\Leftrightarrow R + C = \emptyset$
  - polynomial time via LP
- rank $r \Leftrightarrow R + C$ has rank r
- rank $1 \Rightarrow$ poly. time
- rank $3 \Rightarrow$ PPAD-complete [Mehta 2014]

$\boxed{\text{OPEN}}$: ε-Nash: $\leq$ additive ε incentive for player to change
- no FPTAS
- quasi PTAS: $n^{O(\frac{\lg n}{\varepsilon^2})}$ time

Not covered (but in slides): more PPAD reductions

<u>PPA</u>: "if a graph has a node of odd degree then it must have another" (Handshaking Lemma)

- undirected version of PPAD
- Odd Degree Node:
  - one circuit $C: V \to$ sets of 2 vertices
  - edge $(u,v) \Leftrightarrow v \in C(u) \wedge u \in C(v)$
  - $\Rightarrow$ max. degree 2
  - if given node has odd degree, find another one
  - PPA-complete
- crucial here & in PPAD that we ask for <u>some</u> other vertex, not other end of <u>same</u> path ~ else not in FNP
- Smith: given Hamiltonian cycle in <u>3-regular graph</u> (explicit), find another one (always exists [Smith] [Thomason 1978])
  - $\in$ PPA
  - $\boxed{\text{OPEN}}$: PPAD-complete?

<u>PLS</u>: "every directed acyclic graph has a sink"
[Johnson, Papadimitriou, Yannakakis 1989]

- Find Sink: (n-bit node ids.)
  - circuit $C : V \to 2^V$ → sets of vertices
  - circuit $F : V \to \mathbb{R}$
  - edge $(u, v) \Leftrightarrow v \in C(u) \wedge F(v) > F(u)$ ⇒ DAG
  - <u>goal</u>: find a sink (x s.t. $F(x) > F(v)\ \forall v$)
- PLS = {FNP problems reducible to FindSink}
- Local Max Cut:
  - given a weighted graph
  - find a partition $V = V_1 \,\dot\cup\, V_2$ such that can't move any one vertex $V_1 \leftrightarrow V_2$ to increase size of cut $E \cap (V_1 \times V_2)$
  - PLS-complete [Schäffer & Yannakakis 1991]
  - easy for unit/small weights

<u>PPP</u>: "if a function maps n elements to $n-1$ slots, then there is a collision" (Pigeonhole Principle)
= {FNP problems reducible to Collision}

- Collision: given circuit $C : V \to V$, find ① x s.t. $C(x) = 0^n$ OR ② find $x \neq y$ s.t. $C(x) = C(y)$

<u>Relationships</u>:

$$\mathrm{FP} \subseteq \mathrm{PPAD} \subseteq \mathrm{PPP} \subseteq \mathrm{FNP}$$
$$\mathrm{PPAD} \subseteq \mathrm{PPA} \subseteq \mathrm{FNP}$$
$$\mathrm{FP} \subseteq \mathrm{PLS} \subseteq \mathrm{FNP}$$