6.854J / 18.415J Advanced Algorithms
Fall 2008

18.415/6.854 Advanced Algorithms

Problem Set 5

1. Consider the linear programming relaxation of the vertex cover problem seen in class.

$$\text{Min} \sum_{i \in V} w_i x_i$$

subject to:

$$x_i + x_j \geq 1 \qquad (i,j) \in E$$
$$x_i \geq 0 \qquad i \in V$$

   (a) Argue that any basic feasible solution x of the above linear program must satisfy $x_i \in \{0, \frac{1}{2}, 1\}$ for all vertices $i \in V$.
   Hint: given a bfs x, consider the vector y defined by $y_i = x_i$ if $x_i \in \{0,1\}$, and $y_i = 0.5$ otherwise.

   (b) To solve the linear program to optimality, we can therefore restrict our attention to solutions satisfying $x_i \in \{0, 0.5, 1\}$. For this purpose, consider the bipartite graph obtained by introducing two vertices say a_i and b_i for every vertex i, both of weight w_i, and having edges (a_i, b_j) and (a_j, b_i) for every edge (i,j) of the original graph. Show that the minimum weight of any vertex cover in this bipartite graph is exactly equal to twice the value of the above linear program. Also, how can you extract the solution of the LP from the vertex cover in the bipartite graph and vice versa?

   (c) Show that the problem of finding a minimum weight vertex cover in a bipartite graph can be solved by a minimum cut computation or a maximum flow computation in a related graph.

2. Consider the 2-approximation algorithm seen in class for the generalized Steiner tree problem (we are given a set T of pairs of vertices and cost on the edges of a graph, and the goal is to find a subgraph (a forest) of minimum cost in which every pair of vertices in T is connected).

   (a) Argue that this problem is a generalization of the minimum spanning tree problem.

      i. Does the algorithm seen in class produce a minimum spanning tree in that case? If so, prove it; if not, give a counterexample.

      ii. Is the value ($\sum_S y_S$) of the dual solution y constructed equal to the minimum spanning tree value? If so, prove it; if not, give a counterexample.

(b) Argue that this problem is a generalization of the shortest $s - t$ path problem (in an undirected graph with nonnegative edge weights).

   i. Does the algorithm seen in class produce a shortest $s-t$ in that case? If so, prove it; if not, give a counterexample.

   ii. Is the value $(\sum_S y_S)$ of the dual solution y constructed equal to the shortest path value? If so, prove it; if not, give a counterexample.

3. We would like to design an approximation algorithm for the following problem. We are given an undirected graph G with cost c_e for every edge e, 2 disjoint sets A and B of vertices of the same size, and we would like to find a minimum cost set H of edges such that in every connected component of H, we have the same number of vertices of A and B (so for example a matching between A and B would be one possible solution, and a spanning tree would be another).

   Show that the approach used to design the 2-approximation algorithm seen in class for the generalized Steiner problem can be applied here to get a 2-approximation algorithm as well. Do not reprove everything, but state and prove everything (in the algorithm and/or in the proof) that differs from the case of the generalized Steiner tree problem seen in class.

4. Consider the maximum weight matching problem in a (non-bipartite) graph $G = (V, E)$. More precisely, given a non-negative weight w_{ij} for each edge $(i, j) \in E$, the problem is to find a (not necessarily perfect) matching of maximum total weight. Consider the following greedy algorithm: start from an empty matching and repeatedly add an edge of maximum weight among all edges which do not meet any of the edges chosen previously. Stop as soon as the matching is maximal (i.e. no other edge can be added). Let M_G denote the greedy matching and Z_G its cost. You are asked to show that the greedy algorithm is a 2-approximation algorithm.

   Show that the following linear program gives an upper bound Z_{LP} on the optimal value Z_M of the maximum weight matching problem.

$$
\begin{aligned}
\text{Min} \quad & \sum_{i \in V} u_i \\
\text{subject to:} \quad & \\
& u_i + u_j \geq w_{ij} \qquad (i, j) \in E \\
& u_i \geq 0 \qquad i \in V.
\end{aligned}
$$

   From the greedy matching M_G, construct a feasible solution u to the above linear program and show that its value is $2Z_G$. Conclude that $Z_G \geq \frac{1}{2} Z_M$.