PROFESSOR: All right, let's get started. If you haven't already, there's two handouts on the left, and you should take two pieces of paper. So we'll doing some actual origami folding. We'll be folding 6.849 today, just like this. It'll only take us eight hours or so. This is the Jenny and Eli folding you've seen on the poster. Pretty awesome. No, we'll be folding letters more like this.

So thanks everyone for giving so many cool questions and comments, and the feedback is really helpful. I didn't know what I was going to cover this class. I had too many ideas, and your questions really helped narrow it down into exciting thing. So the structures going to be, I'm going to go through questions, so it's a funny kind of interactivity where there's a whole day in between where you ask the questions and where I answer them. But feel free to ask more questions, follow-up things. But there's a lot already here, so it should be fun.

These are not questions but funnier comments. If you like double rainbow jokes, this lecture happened a couple months after the double rainbow fiasco. There's a lot more, as I recall. It was a running theme throughout the whole semester, so look out for that. I know this is entertaining. I'm getting used to listening to myself at double speeds.

This class definitely is nice in the way that with a fairly simple technique you can prove a very powerful theorem. Sorry, I just remembered I need to push a different button here. That's hopefully a theme throughout the class, but definitely especially nice here.

It is cool, though, strip folding, it was an open problem for a couple years, at least, of no one thinking, how do we fold any shape? That seems really tough. The point of the strip folding approach is that once you have the right idea, to start with a really long rectangle, somehow it becomes easy. There's still a lot of details in getting that to work, but it's kind of neat how that works out.

Now we get to actual proposals. So folding practice, I was planning on doing this, but-- this is an explicit comment to that effect. So we're going to fold some letters of the alphabet. You have in your packet instructions for making the individual digits six, four, eight, and nine, which are all pretty easy. You also have a diagram for this crazy design called Typeset. This same folding can make any letter of the alphabet and any digit.

So just to show you what they will look like these are my foldings of six, eight, four, and nine, according to the first set of diagrams. And then this is my folding of, I think the number six, out of Jason Ku's design. So if you wanted to reconfigure it, to-- I guess eight is kind of hard-- let me do four. OK, so for four we've gotta fold this guy under. Fold this here, this tab, I think goes back here.

So the advanced origami folders in this class can definitely do Jason Ku's design, but it takes a little while. There, I've got a four. Got it? A little hard to hold in position, but, at least in theory, it will make any letter all out of one folding. You just have to move all the tabs around. So it's kind of neat, but it takes at least half an hour or so to fold that, unless you're really fast.

So I would recommend-- pick one of these. Work in groups if you like. If you want you can form a group of four, and make six, eight, four, and nine. Follow the diagrams. This is an exercise of following diagrams, one of the other. We'll only have time to maybe make one digit each, but have fun with it. And if you have questions, raise your hand.

I can tell you, the first step in six, eight, and nine, is to make an eight-by-eight grid. There's a lot of ways to make an eight-by-eight grid, but an easy one is shown here. So you take your sheet. You fold the bottom edge to the top edge-- I think they want to do it white side up-- so they're all valleys. We align those edges. You'll get a nice bisector.

Then you repeat, folding the bottom edge to the middle. And once you do that, to save a little bit of time, you could then fold that new bottom edge to the middle again. That will do eighths on one side. It's a little bit inaccurate, because you're folding through two layers-- but it's a little faster. Time is of the essence.

Then you do the same thing on the bottom, and you'll get eighths in one dimension. But then you have to fold it in eighths in the other dimension.

Once you have eighths, it's like three steps. It's really easy. For six, eight, and nine-- Four uses a different approach, it's a little more free hand. If you want to be more creative, try the four. You've got to eyeball what looks and feels good for number four.

So out of curiosity, how many people have an eight-by-eight grid at this point? Who wanted to? Cool.

So that's the top half of these diagrams, the eight by eight grid. Then it's mostly folding up over individual edges and some corner faults, but they're all simple folds in this world. So you're folding through, I think, always all the layers. Oh, no. This is only folding through one layer. But these would always fall into the some layers, simple folds category.

So for example, make a six. Fold this bottom edge up. Fold the left three squares over. Fold this corner up. Fold this corner down. And fold this down, this over. There's a really big six.

And these numbers are all pretty much proportioned correctly. The four, you have to be-- it helps to have a reference model of one of the other digits to make it the right height, but it'll end up roughly correct, anyway. The eight's a little bit narrow in this design, slightly narrower than six and the nine. But otherwise they're nice compatible digits.

There's a whole alphabet on the website that's linked from this slide, so you go check it out. Origami club. So it's kind of fun to think about font design and alphabet design. There's actually a lot of origami alphabets out there. This is one of the simplest. It has digits. Anyone have questions? How many people folded a letter, a digit? OK. You can keep folding, but maybe I will continue on. It's going to be a lot of fun crackling. Anyone folded this one? Anyone working on it? A bunch of people, cool. Let me know when you finish. It's kind of fun. It's not that hard.

AUDIENCE: Yes, it is.

PROFESSOR: I wanted to point at the OrigaMIT website. OrigaMIT is the origami club at MIT, and at the top you see a different alphabet. This is a four-fold alphabet designed by Jeannine Mosley who's an MIT alum and came to this class two years ago. And so that's the reference design in 2002. I'm not sure if it has digits, though. At least, the diagrams we found for the letters, do not also have digits.

So I think an interesting challenge is to design a four-fold digit set to complement her letter set. If you're interested, that could be a cool project to work on. Folding design, minimal fold alphabets. You could try a three-fold alphabet, two-fold alphabet.

Also on this website are the meeting schedules. It always Sundays, sometimes 2, sometimes 3 o'clock. You should check it out. And there's a convention coming up, our very own origami convention on October 27th. Jason Ku wanted me to remind you about that coming up. Even if you've never done origami before, other than today, you should check it out. It'll be fun, lots of different sessions, from simple models to complicated models.

This is a cool design by Brain Chan, another MIT alum. It's one square paper folded into the Mens et Manus logo, but, here, instead of the oil lamp, you've got a little origami crane. Cool stuff.

So we proceed on to other questions. This is a pretty simple one. At the top of the notes, it says folding, AKA silhouette folding and gift wrapping, it has a couple of references. So where do those terms come from, is the question. And one answer is it's the title of the paper, "Folding Flat Silhouettes and Wrapping Polyhedral Packages," but that's not the real answer.

So this is the two of us and Joe Mitchell. It's also, I think, the introduction of the term computational origami, but those terms come from earlier references. So in that paper, there's a sentence, classic open question origami mathematics, and we don't really know where it came about. But it was first formally posed by Bern and Hayes in this SODA '96 paper, which we'll be talking about in the next lecture, lecture three.

And this is a quote from their paper, "Is every simple polygon, when skilled sufficiently small, the silhouette of a flat origami?" The point of saying the word silhouette is that when you fold something, like this number six-- there's a whole bunch of layers, and there's a lot of complexity to this folding. By saying silhouette, we just mean, collapse all layers, ignore the coloring, and just take the outline.

So the silhouette of this thing is a rectangle. And, in general, that's sort of the transformation to throw away the complexity of the folding, and say, I just care about the shape. Can I get the desired shape?

There's some other interesting questions here, though, which haven't been fully addressed. How many creases are necessary to fold? We'll actually get to that in a later question. How thick must the origami be? The strip method shows this if you start from a rectangle of paper, the number of layers can be very small. I think two or three is enough for all the gadgets that we use, probably three . If you start from a square, though, we don't know the answer to that question. And in practice, folding through many layers is tough. So that is a silhouette problem.

The gift wrapping problem, the motivation is you have a weird-shaped gift. You want to wrap it with a piece of paper. And this is posed to us in a talk by Jin Akiyama at a Canadian geometry conference. Jin Akiyama is a really cool guy. He has had for many, many years a mathematics TV show in Japan. And he's known throughout Japan, because everyone in school watches his videos. And it covers really interesting mathematics.

Some of the results in this class are actually in his videos as well. It's mostly in Japanese, so it's a little hard for most of us watch. But there's some subtitled versions, and they're really fun. Maybe we can have a movie night, and watch one of them, if I get permission. So there isn't a great reference for that. I mean, he's written some papers about different kinds of wrapping problems, but mostly it was this talk that he gave in 1997, which is when I was just starting out in computational geometry. That's where the terms come from. A lot of words.

Have you ever actually folded a model using this method of zigzagging and folding with the strip? Any real or sensible or pretty origami models, or is it purely for the sake of universality? My knee jerk reaction was, no way is this practical. This is just for universality. And the point of this theorem has always been, in my mind, to prove that everything is possible, but then the challenge is to find good foldings, for some notion of good.

But, actually, there are a bunch of examples of strip folding, not a lot of different folds, not terribly many gadgets, but there's some cool things, especially with strip weaving. These are just a few examples of woven colored strips. You can make fun things like Space Invaders. You could weave together baskets, and wrap your packages, and so on.

This is a little bit more origamic. So there were no real folds in there, except at the edges of the cube. This one is a modular origami. It involves a bunch of different folds to get all the pieces to lock together. Modular origami means you have a bunch of identical pieces. They kind of weave together through folding, and then you can make a nice little crown.

This is a very classic model. You've probably seen at some point. Here that the paper's slit, and then it's woven. But there's some folds down here. Not a lot of folding, but strips are pretty neat. You can definitely use them in all sorts of different designs.

Here's some more sculptural designing models from taking strips of paper. This has no glue in it, so I think there's more strips at the end locking this together. And this guy, Zachary Futterer, took a bunch of these kinds of units and started weaving them together to make really complicated shapes. So you can definitely do cool things with strip folding.

And another common one around these days is taking gum or candy wrappers, and folding them down into little strips and weaving them together to make handbags and other things. This has become kind of a fashion trend over the last few years. So those are things you can do a strips.

We have used it in one paper where the goal is actually to be efficient and use a small piece of paper, and not just prove some universality result. This is in our paper, folding a better checkerboard, which we'll talk about in two lectures, if I recall correctly, in more detail. But this is sort of a baseline. This is not the better method that's developed in this paper, but it's the starting point.

You take a square, so this actually starts with a square. You do this pleating. And this is with bi-color paper. It's dark on one side, light on the other. You get this strip of squares in color pattern. And then you take that strip-- a huge number of layers in the middle, so it's not super practical, but it's actually pretty efficient in terms of how big a square you start with.

To make an n-by-n, this is obviously not to scale. You need more squares here in order to make this thing. And then you just snake your path back and forth. You could use turn gadgets, or here we're just using 45 degree folds. And this is pretty close to what was believed to be the best way to fold a checkerboard, and then this paper shows how to do a factor of two better. So we'll talk about that later.

But there are some uses for strip folding. This is a little bit theoretical, but it's actually pretty competitive against the best n-by-n checkerboard foldings in the origami world, like the one I showed last class. So that's practicality of strip folding.

Next question's more about strip folding. There are a couple things that are in the lecture notes, the handwritten lecture notes, but were not even mentioned in the audio part of the lecture. So a few people asked, what are these things? Said pseudopolynomial upper bound. Pseudopolynomial is a fun term. Let me tell you a little bit about it. It's from the algorithms world, but even a lot of algorithms people don't know it. So let me tell you.

So maybe first I should tell you about polynomial. In general, what these terms are about is measuring how fast an algorithm is. So the idea is you plot, conceptually, n, this is the problem size. If you wanted to fold an arbitrary polyhedron, the one way to think of the problem size is the number vertices, edges, and faces. Just the total number of things you're given as input, and then your output is whatever. But n is supposed to be the input problem size.

And then on the y-axis, you want to plot the running time of your algorithm. So this is how long it takes to compute the way to fold your square paper into your desired shape, and generally this is going to increase. And the question is, does it increase in a reasonable way or in a crazy way that goes exponentially high?

So you want to know, how does the running time grow with n? Polynomial is a sense of good growth, and it just means you grow, like, n to the c, where c is some constant. So ideally you'd have n-- or maybe you have n squared, or n cubed, or n to the fourth-- all these are considered good running times.

Not quite as good as polynomial is pseudopolynomial. And I would conjecture, for this problem of folding an arbitrary given polyhedron, you cannot achieve a polynomial number of folds, let's say. So there are two things we could measure here, the running time of the algorithm-- we could measure, actually, three things-- we could measure the number of folds you make, the number operations you do on the paper, and a third thing would be scale factor. How big a square do I have to start with in order to make a desired polygon?

And pseudopolynomial means n times r to the c. What's r? R is some geometric parameter, geometric ratio, in the input. And in particular for this problem what makes sense for r is basically the longest length divided by the shortest length. This is typically what r refers to. This will come up in later lectures as well.

So for example, you take your entire shape, you measure the diameter of the shape, the two farthest points. That's your longest length. Shortest length would be-- maybe you have a triangle, something like this, in the target polyhedron you want to make. This would be your shortest distance. This is actually called the minimum feature size in computational geometry or the minimum altitude of any of your triangles. OK, so that's just some number.

And you can have a triangle, which is super, super narrow. And so it's this ratio r could be arbitrarily large even though you only have three vertices, three edges, one face. So n and r not necessarily comparable, so that's why in pseudopolynomial we put them both together, and then we raise them to some constant power. That's a pseudopolynomial running time.

So the question that's being posed here is, can you get a pseudopolynomial upper bound, and can you get a pseudopolynomial lower bound? And it doesn't say for what, but it's for all three problems-- running time, number of folds, scale factor. And not all of these are open. So in the original paper, there's this theorem that says lots of things-- you can fold anything. Then it says here, the folding requires a number of folds, polynomial of n and the ratio, r.

So it already claims it there is a pseudopolynomial bound on the number of folds. It doesn't say what that pseudopolynomial bound is. Is it n times r? Is it n plus r? Is it n times r squared? I would guess one of the first two-- n times r or n plus r. So that's the upper bound question. Maybe we can work on this in a problem session.

A lower bound question is, do you prove that you need some dependence, both on n and r? Which I would guess is pretty easy. If you want to take a square and fold it down to a really, really skinny triangle, I think you need at least r folds, roughly. And similarly you should need at least n folds, so there should be a lower bound like n plus r. But none of these have been written down explicitly, so that's what those open questions are.

Then there was another slide, six, which was completely uncovered. Stop me if there are questions. I should maybe take a brief moment to breathe.

So the next part of the lecture notes ask about seam placement. So seam placement has the following kind of issue. When you fold, like this number-- is this a six? This is a nine. Fold this number nine. In addition to seeing the color pattern, if you look closely there's also these kinds of seams. This white square is not just a white square. You can see on the top layer this crease line. And here there's a seam, here there's a seam. These are like visible lines. Of course, you have to have seams at the color transitions, but there's other seams as well.

Maybe you want to minimize the seams you want to get, you want to place the scenes in a cool pattern. When you fold checkerboards, there's a such thing as a seamless checkerboard, where every square is a whole square paper. There's no visible crease lines on the top layer.

So this is an extension of the universality result, to also get sort of universal seem placement. And what the original paper proves is that you can place the seams however you want, provided the seam regions, the regions between the seams, are convex polygons-- which is almost always the case.

You look at a typical model-- here, the seam regions are all rectangles and triangles. So you could achieve exactly this seam pattern if you wanted it. You could also say, oh, here's a nice rectangle. I'll make that a seam region. Here's a nice rectangle, I'll make that same region. But you could not make the entire number nine here a seam region, because it's non convex. At least, you can't do it with this technique.

We don't know, necessarily, whether this is possible by some other folding. I would guess no, but it is possible to make some non-convex seam regions. For example I could take this page and fold the corner over, and now I've got a non-convex seam region here. So some non-convexing regions are possible.

Open question is-- if I give you a polygon-- we know every polygon's possible. Now I give you a polygon and I subdivide it into seam regions. Which of those are possible? Not everything is possible, I'm pretty sure, though I'm not sure we have a proof of that.

Some things like this little heart shape are possible. Characterize. This another cool possible problem for problem session. Questions about that? I have a little bit about the proof of how this is done. If you wanted to just do convex regions. So the general approach here is you want to visit all of the regions in some order. This is called the tour. It's pretty easy to just-- I mean you're allowed to visit regions more than once, so you just keep going, keep trying to visit some unvisited seam region.

When you visit a seam region, it's a convex polygon. So what we're going to do is make our strip fairly wide, actually, wide enough to completely cover that seam region. And then at this moment, we basically need to turn to do the next one.

We know how to change the direction of the strip using a turn gadget. Then we have to change the width of the strip. Maybe it needs to be wider. Maybe it needs to be thinner.

And then we need to shift the strip one way or the other. So if we just end here, we turn, we might be misaligned. We need to shift it over, expand it, then do the next one, then turn, then shift it over and set the right width. Keep going like that. OK. That's pretty messy and complicated, but you can do with these two gadgets.

Strip width gadget-- you take a strip, and you could make it anywhere between 1/2 and 100% of its original width. So the idea is you start with a really wide strip, wide enough to cover all the polygons. Then you do this gadget and keep shrinking it by half until it's roughly the right size. And then when you're almost correct, you shrink it by a little bit more-- here shrinking to a third. And then you get your shrunken strip, and it happens right at the line that you specify.

So you can basically on a dime shrink your strip, and then by doing the reverse can grow it back. This is maybe not with simple folds, though. And then the other gadget is a shift gadget, where you're at this position and you want to shift up. So that's pretty easy, you just do two turn gadgets. So that's at a high level how making a desired seam pattern works.

Go on to the next question. A lot of people asked about this-- and this is an open problem that I mentioned orally. It's not written in the notes-- which is, can you actually do the things we said we can do with simple folds? So can you get a universal folding of a polygon, two-color pattern polyhedron, using simple folds?

And I thought it'd be fun to actually work on this, here, live, because I think this is an easy problem. And there's a bunch of possible answers, and there are even two suggested ideas from the comments field.

So let me just remind you of the issue, what's happening, and then I need your input, what's going to work here. So general picture for the strip method was we do one triangle, we end here, then we do a bunch of folds like this, and then we end here, maybe, and then we zigzag. And the trouble is we've already made this triangle over here.

When we make this triangle, we have this excess stuff, which I haven't drawn very accurately. If you recall, it looks something like that. Or maybe even more like that if we use right angle turn gadgets. And then we want to fold it underneath and we're doing that, the way I said, with height gadget mountain folds. But the model's simple folds, which I should make more explicit.

You're not allowed to collide during the motion. The idea with a simple fold is that you should be folding along one line segment, and you should fold by-- at least the model that we defined back in the day, I'll talk more about where this notion comes from-- you fold by plus or minus 180 degrees, which means after you do the fold, you'll be flat again. And no collision during the motion.

If we folded this triangle, and then we folded this one, and this stuff is on the top-- we can't mountain fold. That's not considered a simple fold. Now one proposal is, could we just make the next triangle underneath the previous one? A different proposal is valley fold. These are actually different proposals, because, especially for two color patterns, it'll make a difference.

If we valley fold here there's going to be some junk on the front side, especially if you want to get a desired seam pattern. But maybe we'll leave seam patterns for later. If you want to get a color pattern, you might reveal some wrong color when you do that valley fold.

So I haven't really thought about this idea yet. I think it might be good. The idea there would be-- so you've already made this triangle. You mountain fold everything. Now when you go and you do these zigzags, you want to be underneath everything that you've done.

AUDIENCE: Can you fold the one you've already made, like as a unit, over the side? And then make it and then just put it back on top?

PROFESSOR: OK. Different idea is you basically fold this out of the way, do this thing, height gadget, and then fold it back. Maybe. I've wondered about that, too. Is it's going to get a little challenging, though. In general, there's a huge set of triangles, so unless you can like go far away, make your triangle, and then plop it down-- maybe it's possible.

I guess we can pursue that idea, but maybe first we should exhaust the easier ideas. I mean, that is definitely plausible that that's possible. Doing that with simple folds and not leaving any garbage is going to be a little challenging, but it might be doable.

This to me is the simplest idea, so we should first see if it works. Does anyone see problems with this plan? I have some strips. We could think about what it means to be doing a turn gadget underneath here.

So I don't know quite about, well, let's suppose we are already here. And now maybe I do some as the turn gadget goes to mountain folds. Then at 90 degrees, and then a valley fold like that. And then I do a mountain fold. I think they might be OK, because turn gadgets start with a mountain fold. So if you're underneath everything, that's going to avoid collision. And then the valley fold brings it back.

Again, we're using just the space below everything that we see. So then we make that strip, we keep turning around, and then later on we're going to mountain fold this behind, somehow, to meet this edge. Seems OK. Yeah? Question?

AUDIENCE: I'm sorry. I'm confused about something. When you're doing the turn gadget, while the paper is above the first triangle you go over, you're doing a bunch of turns where you're taking a piece of paper up another one, folding just that part Is that a s--

PROFESSOR: Going from one triangle next. Yeah. So we could think about that, too. It's very, you're right. I mean, I'm just looking at the turning around part for making a single triangle, but there's also the turn gadget going from here to here. It's actually slightly more complicated than we've covered in class, but not really much. It's just a slightly generalized turn gadget. So you're coming here, you basically want to turn around.

Let's just think about a way to do it. You could imagine first doing a turn like this. It's not exactly a pure turn gadget. And then turning around to get to next place-- this is really hard to do on a blackboard. Strips just tend not to stay together well.

OK, now we're going parallel to the correct direction, and then we turn back and forth. But each of those is just using turn gadgets. As long as a turn gadget works fine, a turn gadget is going to be a mountain fold, which is going to go behind everything, and then a valley fold to bring it to a desired direction. Those are all using everything behind the board. So it seems like all those operations are OK.

Now we should also check the color reversal gadget, which-- it's funny thing. I remember everything I did before 2000 or so, so I still have memorized the color reversal gadget, at least I think so. I should probably color this piece of paper, so you can see the colors change. I don't remember anything I've done since 2000, but anything up to 2000 I'm OK. This is 1998, I think.

I remember folding lots of-- this is ticker tape, they use it for or not ticker tape, but they use this for like adding machines. So I think it's a 90 degree mountain fold, then you fold up like that, and then you fold back down with a mountain fold like that. You get a color reversal. And all of those folds we're working behind my plane here, so should avoid collision with everything.

I think you could do color reversal and turn gadgets behind. And so the suggestion works. Who made this suggestion? Good idea. Unless there are any objections, I think that will work. I had a different plan, which was to use this second idea and set up the turn gadgets so there is no-- when you fold this with the valley fold, there is no ugly colors. So you could maybe modify the turn gadget to be completely solidly colored on both sides, but I think this is much easier. It's probably why I didn't write it down the notes, but I'm not sure. Yeah?

AUDIENCE: I think you could also just do a thing where you just take it far away, because you have a really long strip. So you can just take that strip, and go to where there are no things, fold it, and then take it out and put it on top.

PROFESSOR: OK. How do you do that last part?

AUDIENCE: Accordion fold it with the right lengths.

PROFESSOR: So you have to do it with simple folds. That's the main, that's the challenge. So the idea is you're way out here, you have a triangle out here, which you want to bring over here. Maybe I should do it like this. You could do something like this, so now the triangle's over here. Then maybe you want to go almost all the way here and then fold it back, and then fold it forth, and back, until you get your triangle exactly where you want it.

AUDIENCE: Figure out from the base--

PROFESSOR: It seems plausible.

AUDIENCE: --where you need to fold it.

PROFESSOR: My only concern would be when you do this thing, there might be a little corner. Depends how you fold this thing, and then you've got to hide that corner. And if there's triangles all around here, there may not be room. I mean, maybe if there's triangles all around here, it's OK to have that corner, but maybe the triangles are different colors. So I do believe that should be possible, but I think it is a little bit more complicated because you have to hide one last piece after you get in position.

Anyway, I think there are at least three ways to solve this problem. Yeah?

AUDIENCE: Are you allowed to unfold?

PROFESSOR: Are you allowed to unfold? That's a good question. I don't remember whether the original model says whether you're allowed to unfold. So if there are two versions, simple folds and unfolds, or just simple folds. I don't think we actually said unfolding is allowed. Though we're definitely thinking about at some point, it's probably not in the model as defined. Any questions?

What's next? This is the paper that introduced simple folds. It's called "When Can You Fold a Map?" because it originally was motivated by map folding. And it had a bunch of reasons for introducing simple folds, among them is this quote, which if you watched L1 was in there. I think the easiest way to refold a roadmap is differently and her goal was to make it easier to refold your roadmap correctly. So here's one quote from that paper as motivation. So it's origami motivation.

But we're also wondering about applications, like sheet metal bending, cardboard folding, things like that where you want to manufacture things using a machine. And while origamists can do complicated folds, non-simple folds, to make art work, in practical manufacturing, you want to have the simplest possible machine.

So if you can get away with just simple folds, as defined here, that would be great. Now you don't really need, some of these are maybe artificial. You probably don't need the 180 degree condition, because most of the things you want to fold aren't flat. We introduced that just to keep things simple mathematically. But you'd like to fold along just one segment at a time, ideally. You definitely don't want collision. You don't want material to hit things.

Whereas in origami, you can do tucks, you can do things that are not simple folds. That's a lot harder with a machine that doesn't have any feedback. So here's a very simple machine. This is a brake folder. We actually have a brake folder in CC, although this one is Electrabrake, so this has an electric assist.

So the idea is you slide your sheet in, and you hold here. You pull up, and, in this case, he's bending to a 90 degree angle. You can adjust it to different angles and so on. There are lots of automated machines, it's a little hard to get photos and videos of them, but they're based on this principle. Maybe you push in a v, and you end up with the crease in one spot.

And you'd like to just make a sort of conveyor belt with lots of different pushes and pulls and do a bunch of simple folds, basically, except for this 180 degree constraint. And so we're just curious about what's possible by simple folds, and that led us into the map folding stuff, where it's fairly easy to characterize. Other things where it's harder, we'll see in lecture three.

I thought I'd show you some examples of things people make with pretty much simple folds, other than this 180 condition. Out of things like-- this is folding wood. You take a sheet of material. You start bending these parts up, and you can make a little chair, a little table. And you could fold it back when you're not using your living room. You can hide everything. So you could imagine also having multiple sheets, and sometimes your room is a living room, other times it's-- whatever furniture you need, you just unfold the appropriate thing. That's the vision.

Here's a cute little folding chair. There's a huge number of folding chairs, but this one is pretty much simple folds. The one thing I'm not sure whether it falls under simple folds is this fold. You do fold along one line, but it's in two different pieces. I'm not sure we'd call that a simple fold. There's, of course, lots of slits in the material here. But of course it has all the same advantages of simple folds. This is easy to execute one step at time.

Here's some more complicated design. Some of these are computer renderings. Some of these are real. Again, taking furniture out of flat walls. And here's some table designs. These are sheet metal. I like this one. It's very simple. Take the square of sheet metal, put in some slips do, some very simple folds-- boom, you've got a table. This one's also pretty simple.

Again, here we're folding along one line, but it's in two different pieces. So is that a simple fold? It's definitely harder to build such a machine, but it's doable.

Here we have something that's definitely not a simple fold, but it's also fairly easy to execute. Using a roller, you can kind of curve one segment. I mean, when you go to reality, you can change the model all sorts of different ways and still have something practical. And no one rule set is gospel. But mathematically we have to hone in on at least one model at a time, and then we can see how changing the rules changes what you can make.

OK, next question. This is actually about the definition of simple folds, so it was probably answered already. Is it allowed to bend the rest of paper to get it out the way and avoid collision? The answer is no. In simple folds, at least, you're only allowed to move that one segment.

We have actually lately been thinking about a different model, where you do allow this, but simple folds, you can't move other parts. You can just move the single hinge that you're folding. And the end product has to be flat, yes, in our model. Though, it would be interesting to think without this condition, because you're doing 180 degree operations before you do the next one, you'll be flat at all times. 1D or 2D, according to whether you started with a 1D piece of paper or 2D piece of paper.

OK. The second half of the lecture was basically about proving, characterizing flat foldability of 1D segments. And it showed in particular that simple folds are universal, that if you have some mountain valley pattern, and it's foldable at all, if it's flat foldable, it will be flat foldable via simple folds. And in particular using crimps and n folds.

And it was a bit of a messy prove, partly because I've made a mistake in lecture, as you saw. I kind of corrected for it on the fly, but it's maybe not the best written. So I wanted to go through a couple quick examples to make clear all the issues there. So here are the ones I prepared. We can certainly do more if it's still not clear.

So here's a simple mountain-valley pattern, and it's got some long segments and, let's just say, equidistant segments here. Three valleys, then a mountain. So first question is, is this mingling? And then the ultimate question is, is it flat foldable?

So is it mingling? Well, maybe you could answer for me. Just yes or no. 50% chance. So maybe the definition of mingling is not super clear. Let me review it.

So you look at each, I mean, generally of a sequence of mountains and valleys, you look at a chunk of all valleys, then you look at a chunk of all mountains, and chunk of all valleys. Here there's only two chunks-- three valleys and one mountain. And the definition is a little awkward for a single crease, but let's start with the valleys. The point is to check-- for the first segment between two valleys versus the segment just before it, which is bigger? And this is the bad case this is the non-mingling situation, because this thing is bigger than this.

Strictly bigger, the notation we use in the lecture is an open square bracket. So square bracket meant that this is bigger. Round bracket would mean this is less than or equal to this. That's just the definition.

Over on this side of the valleys, this length is equal to this length. So the last distance between two values is equal to the one right after it. And that's a good case, so we write a closed, round bracket.

Then we have a sequence of mountains, and here it's a little confusing, but it's the same idea. So this is the very first mountain. You look at the length right after it versus the length right before it, and this is smaller. And that's a good case. We write an open, round bracket for this mountain group.

And then same thing. Now we're comparing the same two distances, but it's now bad, because this one is strictly longer than this one. So we write a closed, square bracket.

So that's the notation in this example. Any questions about that? So you just have to check-- in general, you have a whole group of valleys-- these are all valleys, or all mountains-- and you want to compare this one versus this one. And square or closed, according to which is bigger. And you want to look at the last one versus the after last one. So that's the notation.

And the point of the proof was to argue that, either if you're going to be flat foldable at all, actually if you're mingling-- mingling meant that for each of these intervals, at least one of the sides was round. That was considered good. So this crease pattern is mingling, because there's two regions. This one has a round bracket. This one has a round bracket. And what we argue is it if you're mingling, which was necessary, if you're flat foldable, you have to be mingling. It's a necessary but not sufficient condition for flat foldability.

If you're mingling, either you have a pattern like this, close round bracket, open round bracket. That's good because this is a crimp that you can do. You see it up here. This is a crimp. You valley fold, mountain fold, and you don't hide anything when you make that operation. Or there's an end fold, which corresponded to an open round bracket at the beginning or closed round bracket at the end.

So here there's a crimp. Let's do the crimp. So when we do the crimp, let's keep this part of paper fixed. So this we go over to here, hopefully, then we valley fold, then we mountain fold, and we keep going from there. That segment is that segment. So we still have this valley. This was a valley we just folded. This is the mountain we just folded. Now conceptually, we just sort of fuse this back into the paper, because those creases are done. We don't need to think about them. The point is, in that region there were no extra creases.

These round parentheses will guarantee there's nothing here, no creases here, here, or here. Could be creases farther away but by these inequalities, that this length is less than or equal to this one, and this length is less than or equal to this one. Sorry, greater than or equal to. Then you know this is OK.

So there's two valleys left. So now we have two valleys, we have a long segment, and a long segment. And this is something that can't be made. Because there's no folded state of this thing, never mind simple folds, because it's going to cross like that. So this is not flat foldable. It's also not mingling, because if you look at these two valleys--

You look at the distance over here. It's bigger than this one, so that's bad, so you have an open square bracket. And this one is also bigger than this, imagine these as fused, and so it's a closed square bracket. And so this group of values is not mingling. So it's not mingling.

So ultimately this pattern we started with is not flat foldable, because one of the things we proved is doing a crimp never changes flat foldability. It's always a safe thing to do. You might wonder, oh, maybe there's some other fold I could to do that eventually works, but we proved crimps are always safe to do. So we did it, and we got stock. That means this was not flat foldable, even though it was mingling.

And so the mingling forever property just means, if it's mingling and then you do a crimp, and it's still mingling, and if you keep doing crimps, and it stays mingling all the way, then you were flat foldable. It's not a very satisfying characterization, but it is a thing. Maybe I'll do one more example where it works, so we're super clear. Whoa, we're out of time, so I won't do another example. All right, ambitious. I gotta work on my timing. There are a

Couple other fun questions here. I would encourage you to read the notes about them. In particular, there's an algorithmic question. How do you actually compute this efficiently? You could do it very efficiently in linear time. So n, where c is one, just n time-- instead of n squared or something-- using a pretty simple idea. Basically just look for the first crimp, do it, and then see if there are crimps nearby, and keep going forward, and you can prove that it takes linear time.

There is this fun question I enjoy thinking about. Can you make any mountain-valley pattern flat foldable by adding creases? The answer is yes. You can think of it as a puzzle. There is one proposed way to do it here. I have another one in the notes. You can think about it.

And the last question, is what does it possibly mean to fold something in four dimensions? How do you imagine it? Hard to imagine, but you can think about it. You have a d-dimensional piece of paper, you fold it through d plus 1 dimensions. If you want it flat folded, it ends up back in d dimensions, and your creases are d minus 1 dimensional. And the rest you just have to visualize.

I have one example folding a solid cube in half in the notes. That's certainly possible. That's not very well studied, and there's lots of interesting open questions about higher dimensional folding.

Any questions before we go? All right. Watch lecture three, and please send your feedback. It was really helpful.