Introduction to Abstract Interpretation

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Course Recap

What you have learned so far

Operational Semantics

- How will a given program behave on a given input?
- This is the ground truth for any analysis

Types

- Annotations describe properties of the data that can be refered by a variable.
- Easy to describe properties that are global to the execution, but only one variable at a time (at least with the machinery we have seen here)
- Properties are fixed a priori by the type system designer
- Actual analysis is cheap
- Annotations can often be inferred

Program Logics

- Annotations describe properties of the state at a given point in the program.
- Easy to describe complex properties of the overall program state, but messy to describe properties that hold over time
- Logic provides a rich language for properties
- Actual analysis can be expensive
- Annotations are hard to infer

?

Some motivation

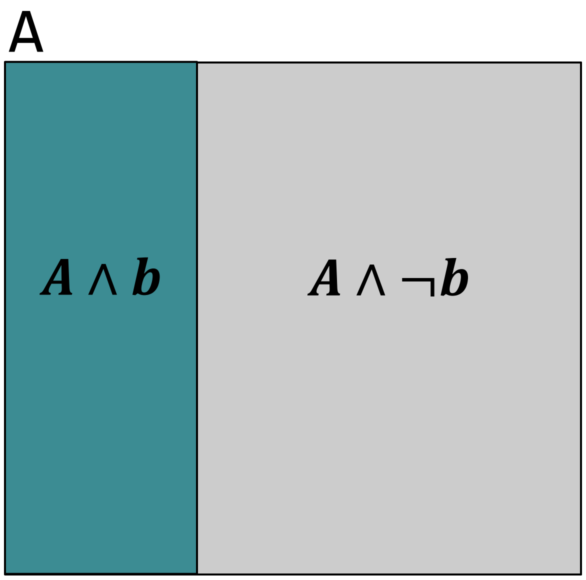
```
{true}
y=0;
while(x<10){
  x = x+1;
  y = y+2;
}
{even(y)}
```

$$\frac{\vdash \{A \wedge b\} c \{A\}}{\vdash \{A\} while\ b\ do\ c\ \{A \wedge not\ b\}}$$

What is the loop invariant?

Intuition:

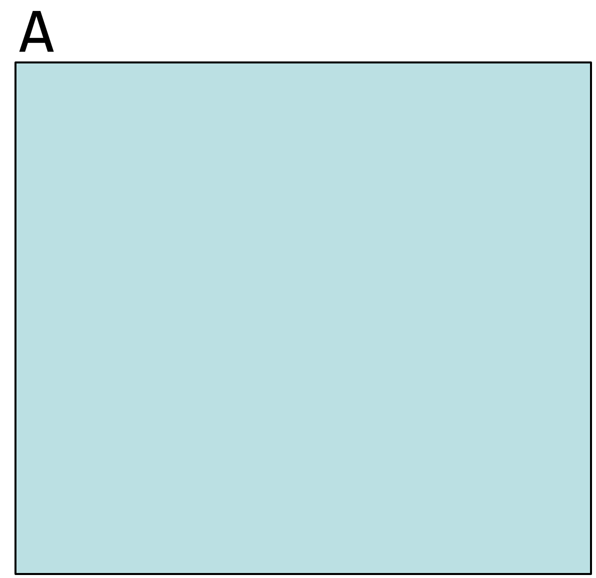
- The loop invariant is a set of states
- C transforms elements in $A \wedge b$ to other elements in A.



Simplifying the problem

```
{true}
y=0;
while(x<10){
  x = x+1;
  y = y+2;
}
{even(y)}
```

$$\frac{\vdash \{A \wedge b\} c \{A\}}{\vdash \{A\} while\ b\ do\ c\ \{A \wedge not\ b\}}$$



This rule is strictly weaker

- Many correct programs can't be proved with it

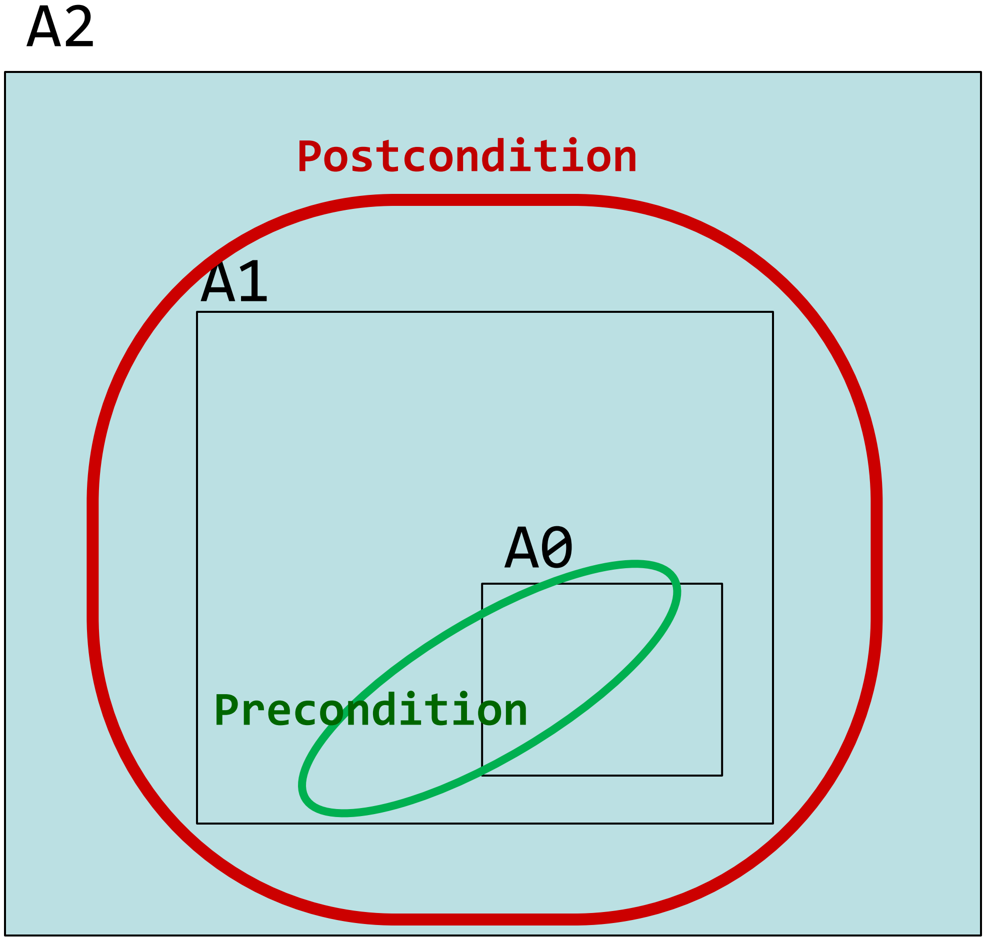
Simpler Intuition:

- The loop invariant is a set of states
- C transforms elements in A to other elements in A.

Discovering the invariant

There may be many candidates for A

- True is always an invariant



Big <=> Weak

Discovering the invariant

We want a set A such that $\dashv \{A\} c \, \{A\}$

- It should be small enough to prove the postcondition (strong)
- But big enough to prove the precondition (weak)

Let $F(P) = \ wpc(c, P) \wedge Post$

- Then what we want is a <u>greatest</u> fixpoint solution of A=F(A)

Convergence properties

- Can we always find such solutions?

Forward vs. Backward

- When is it better to use wpc vs. spc?

Precision

- How do we minimize the loss of precision?

Partial Orders

Set P

Partial order $\leq$ such that $\forall x,y,z \in P$

- $x \leq x$ (reflexive)
- $x \leq y$ and $y \leq x$ implies $x = y$ (asymmetric)
- $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitive)

Can use partial order to define

- Upper and lower bounds
- Least upper bound
- Greatest lower bound

Upper Bounds

If $S \subseteq P$ then

- $x \in P$ is an upper bound of S if $\forall y \in S.\ y \leq x$
- $x \in P$ is the least upper bound of S if
  - x is an upper bound of S, and
  - $x \leq y$ for all upper bounds y of S
- $\vee$ - join, least upper bound, lub, supremum, sup
  - $\vee$ S is the least upper bound of S
  - $x \vee y$ is the least upper bound of $\{x,y\}$
- Often written as $\sqcup$ as well

Lower Bounds

If $S \subseteq P$ then

- $x \in P$ is a lower bound of S if $\forall y \in S.\ x \leq y$
- $x \in P$ is the greatest lower bound of S if
  - x is a lower bound of S, and
  - $y \leq x$ for all lower bounds y of S
- $\wedge$ - meet, greatest lower bound, glb, infimum, inf
  - $\wedge S$ is the greatest lower bound of S
  - $x \wedge y$ is the greatest lower bound of $\{x,y\}$
- Often written as $\sqcap$ as well

Covering

$x < y$ if $x \leq y$ and $x \neq y$

x is covered by y (y covers x) if

- $x < y$, and
- $x \leq z < y$ implies $x = z$

Conceptually,

- y covers x if there are no elements between x and y

Lattices

If $x \wedge y$ and $x \vee y$ exist for all $x,y \in P$

then P is a lattice

If $\wedge S$ and $\vee S$ exist for all $S \subseteq P$

then P is a complete lattice

All finite lattices are complete

Example of a lattice that is not complete

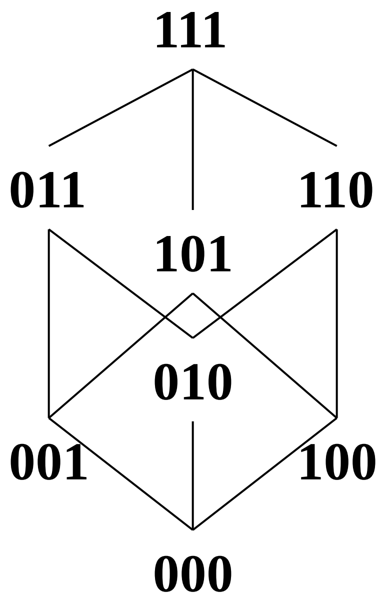
- Integers I
- For any $x, y \in I$, $x \vee y = \max(x,y)$, $x \wedge y = \min(x,y)$
- But $\vee I$ and $\wedge I$ do not exist
- $I \cup \{+\infty, -\infty\}$ is a complete lattice

Example

P = { 000, 001, 010, 011, 100, 101, 110, 111}

(standard boolean lattice, also called hypercube)

x ≤ y if (x bitwise and y) = x



Hasse Diagram

- If y covers x
  - Line from y to x
  - y above x in diagram

Top and Bottom

Greatest element of P (if it exists) is top ($\top$)

Least element of P (if it exists) is bottom ($\perp$)

Connection Between $\leq$, $\wedge$, and $\vee$

The following 3 properties are equivalent:

- $x \leq y$
- $x \vee y = y$
- $x \wedge y = x$

Chains

A set S is a chain if $\forall x,y \in S.\ y \leq x$ or $x \leq y$

P has no infinite chains if every chain in P is finite

Product Latices

Given two latices L and Q, the product can easily be made a latice

$$(l_1, q_1) \sqsubseteq (l_2, q_2) \Leftrightarrow l_1 \sqsubseteq l_2 \; and \; q_1 \sqsubseteq q_2$$

For vectors of L, defining a latice is also easy

$$\langle l_1, l_2, \dots, l_k \rangle \sqsubseteq \langle t_1, t_2, \dots, t_k \rangle \Leftrightarrow \forall_{i \in [1,k]} \, l_i \sqsubseteq t_i$$

Back to our problem

```
{true}
y=0;
while(x<10){
  x = x+1;
  y = y+2;
}
{even(y)}
```

$$x = \begin{cases} \top & \text{Could be odd or even} \\ odd & \text{definitely odd} \\ even & \text{definitely even} \\ \bot & \text{who cares} \end{cases}$$

A latice of predicates:

- <($\mathrm{x} = \bot, even, odd, \top$)>
  - Ex: $\langle x = even, y = odd \rangle \sqsubseteq \langle x = \top, y = odd \rangle$

$$\begin{tikzcd} & \top \arrow[dl] \arrow[dr] & \\ odd \arrow[dr] & & even \arrow[dl] \\ & \bot & \end{tikzcd}$$

What does this have to do with our problem?

Latices and fixpoints

Order Preserving (Monotonic) Function:

$$x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$$

Now, let $x_\perp$ be the least fixed point of $f: L \to L$

- so $f(x_\perp) = x_\perp$

Now, let $x_0 = \perp$ and $x_i = f(x_{i-1})$

- By induction, $x_i \sqsubseteq x_\perp$
- Also, the chain x_i is an ascending chain
- If L has no infinite ascending chains, sooner or later $x_i = x_{i+1} = x_\perp$

Same trick works for greatest fixed point!

- But then you have to start with $x_0 = \top$

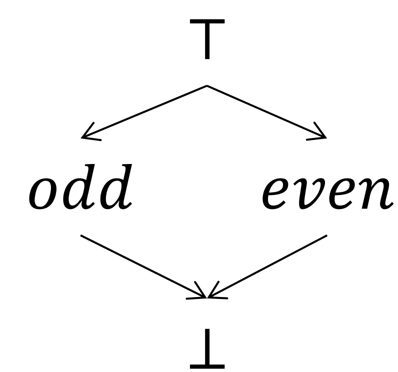
Back to our problem

```
{true}
y=0;
while(x<10){
  x = x+1;
  y = y+2;
}
{even(y)}
```

$$x = \begin{cases} \top & \text{Could be odd or even} \\ odd & \text{definitely odd} \\ even & \text{definitely even} \\ \bot & \text{who cares} \end{cases}$$

A latice of predicates:

- $<(x = \bot, even, odd, \top)>$
  - Ex: $\langle x = even, y = odd \rangle \sqsubseteq \langle x = \top, y = odd \rangle$



We now have a recipe to find a greatest fixpoint solution

- As long as $F(P) = wpc(c, P) \wedge Post$ is monotonic in our latice

Finding a fixpoint

```
{x = ⊤, y = ⊤}
y=0;
while(x<10){
  x = x+1;
  y = y+2;
}
{x = ⊤, y = even}
```

$$x = \begin{cases} \top & \text{Could be odd or even} \\ odd & \text{definitely odd} \\ even & \text{definitely even} \\ \bot & \text{who cares} \end{cases}$$

$$F(P) = wpc(c, P) \wedge Post$$

- $P_0 = \{x = \top, y = \top\}$
- $P_1 = \{x = \top, y = even\}$
- $P_2 = \{x = \top, y = even\}$
- Success!

Complicating things a bit

$\{\boldsymbol{x} = \top, \boldsymbol{y} = \top\}$

```
y=0; t=1;            } C0
while(x<10){
  x = x+1;           } C1
  y = y+2;           }
  if(x=5){
     t=t+2;          } C2
  }else{
     y = t+1;        } C3
  }
}
```

$\{\boldsymbol{x} = \top, \boldsymbol{y} = \boldsymbol{even}\}$

$$\frac{\vdash \{A \wedge b\} c_1 \{B\} \qquad \vdash \{A \wedge not\ b\} c_2 \{B\}}{\vdash \{A\} if\ b\ then\ c_1 else\ c_2\ \{B\}}$$

Relaxed Rule

$$\frac{\vdash \{A \wedge b\} c_1 \{B\} \qquad \vdash \{A \wedge not\ b\} c_2 \{B\}}{\vdash \{A\} if\ b\ then\ c_1 else\ c_2\ \{B\}}$$

$$\begin{aligned} F(P) &= wpc(c, P) \wedge Post \\ &= wpc\big(c1, wpc(c2, P) \wedge wpc(c3, P)\big) \wedge Post \end{aligned}$$

Dataflow equations

```
Big <=> Weak
```
So $A \Rightarrow B$
is equivalent to
$A \sqsubseteq B$

$\{x = \top, y = \top\}$ <-P1
y=0; t=1; } C0
while(x<10){ <-P2
  x = x+1;
  y = y+2; } C1
  if(x=5){ <-P3
    t=t+2; } C2
  }else{
    y = t+1; } C3
  }
<-P2
}
$\{x = \top, y = even\}$ <-P5

$$F(P) = wpc(c, P) \wedge Post$$
$$= wpc\big(c1, wpc(c2, P) \wedge wpc(c3, P)\big) \wedge Post$$

$$p1 \sqsubseteq wpc(c0, p2)$$

$$\boxed{p2 \sqsubseteq wpc(c1, p3)}$$

$$p3 \sqsubseteq wpc(c2, p2) \wedge wpc(c3, p2)$$

$$\boxed{p2 \sqsubseteq p5}$$

$$p2 \sqsubseteq wpc(c1, p3) \wedge p5$$

Dataflow equations

$\{x = \top, y = \top\}$ <-P1

```
y=0; t=1;              C0
while(x<10){           <-P2
  x = x+1;
  y = y+2;             C1
  if(x=5){             <-P3
     t=t+2;            C2
  }else{
     y = t+1;          C3
  }
                       <-P2
}
```

$\{x = \top, y = even\}$ <-P5

$$p1 \sqsubseteq wpc(c0, p2)$$

$$p2 \sqsubseteq wpc(c1, p3) \wedge p5$$

$$p3 \sqsubseteq wpc(c2, p2) \wedge wpc(c3, p2)$$

Dataflow Analysis

General Analysis Framework

- Developed by Kildall in 1973
- Traditionally used for compiler optimization

Frame analysis question as a set of equations on a CFG

Control Flow Graph

$\{x = \top, y = \top\}$ <-P1

y=0; t=1;

while(x<10){ <-P2

  x = x+1;

  y = y+2;

  if(x=5){ <-P3

    t=t+2;

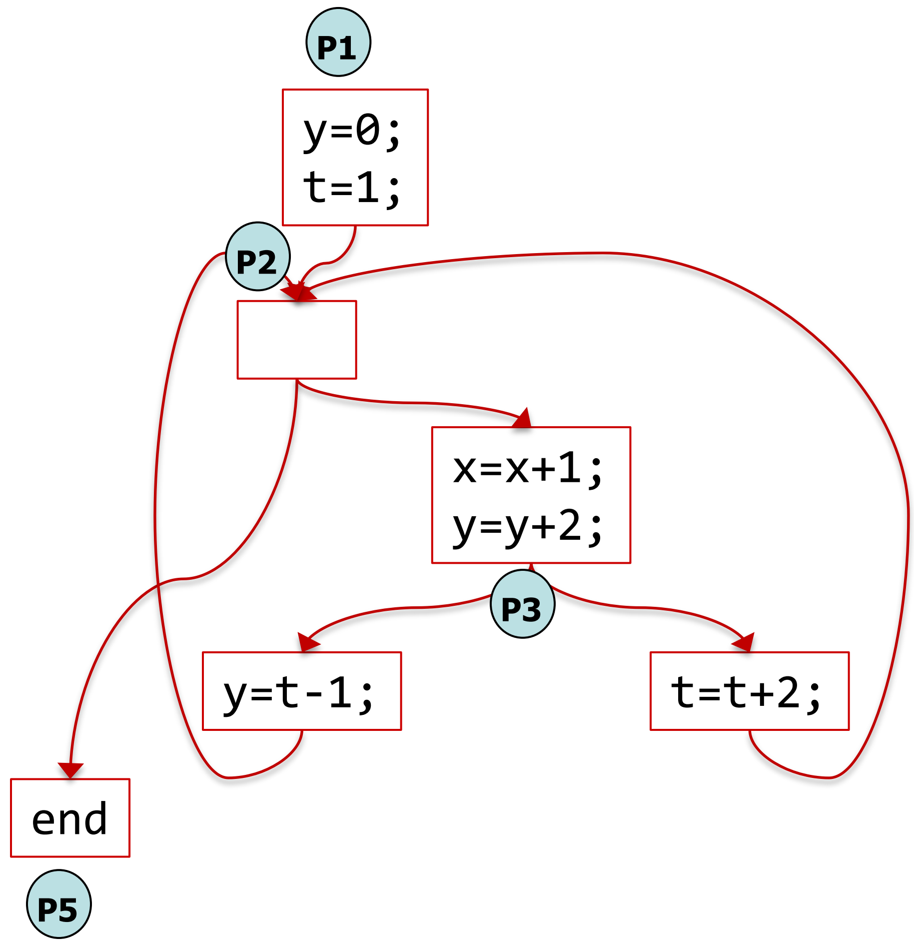
  }else{

    y = t+1;

  } <-P2

}

$\{x = \top, y = even\}$ <-P5



Control Flow Graph

Very general program representation

- Easy to represent unstructured control flow
- Widely used by most program analysis tools for imperative languages

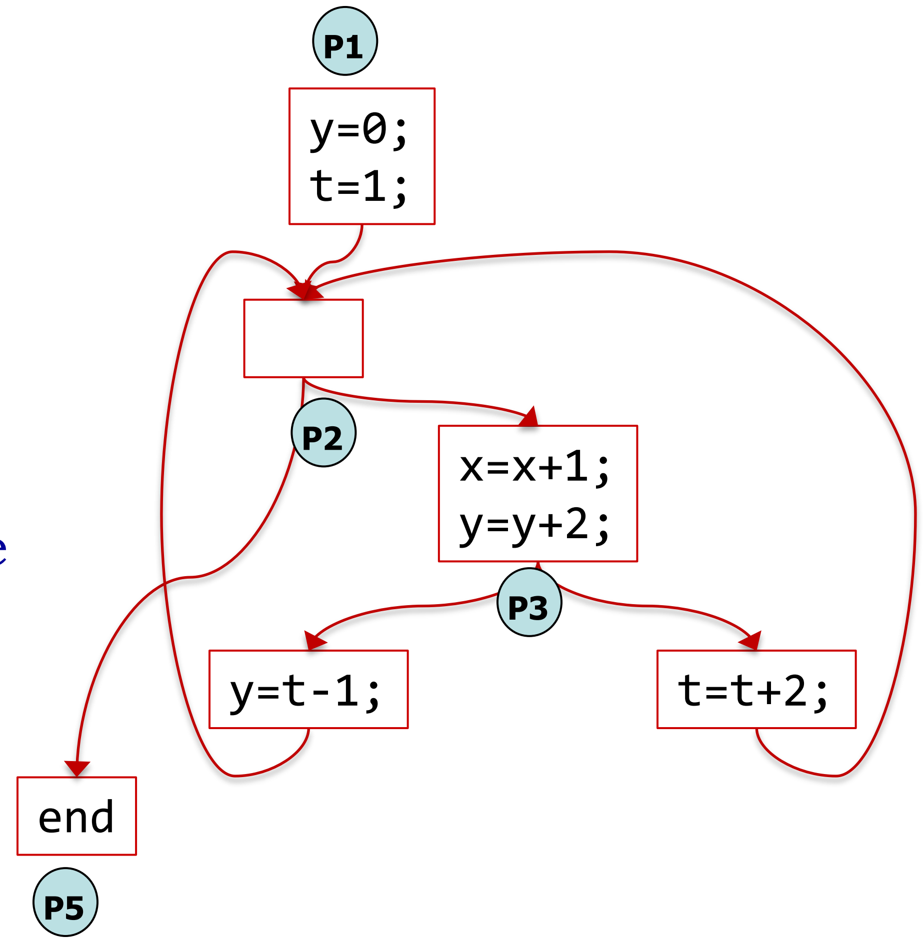
Solution strategy

For every basic block we have an equation of the form

- $Out \sqsubseteq F(in)$
- Use meet (∧) when many edges meet together

We can solve through "Chaotic Iteration"

- Keep a list of nodes to update
- Pick one CFG node at a time
- Update *out* from new *in*
- If out changed, add its children to the list



Computing transfer function

So far we defined it in terms of weakest precondition.

- Or alternatively, strongest postcondition
- Too general and expensive!

We can hard-code a transfer function specific to the lattice

- For finite lattices they can be implemented cheaply in terms of bitvector operations

We can build lattices for arbitrary facts about the program

- Need to make sure our transfer functions are monotonic

Example: Reaching Definitions

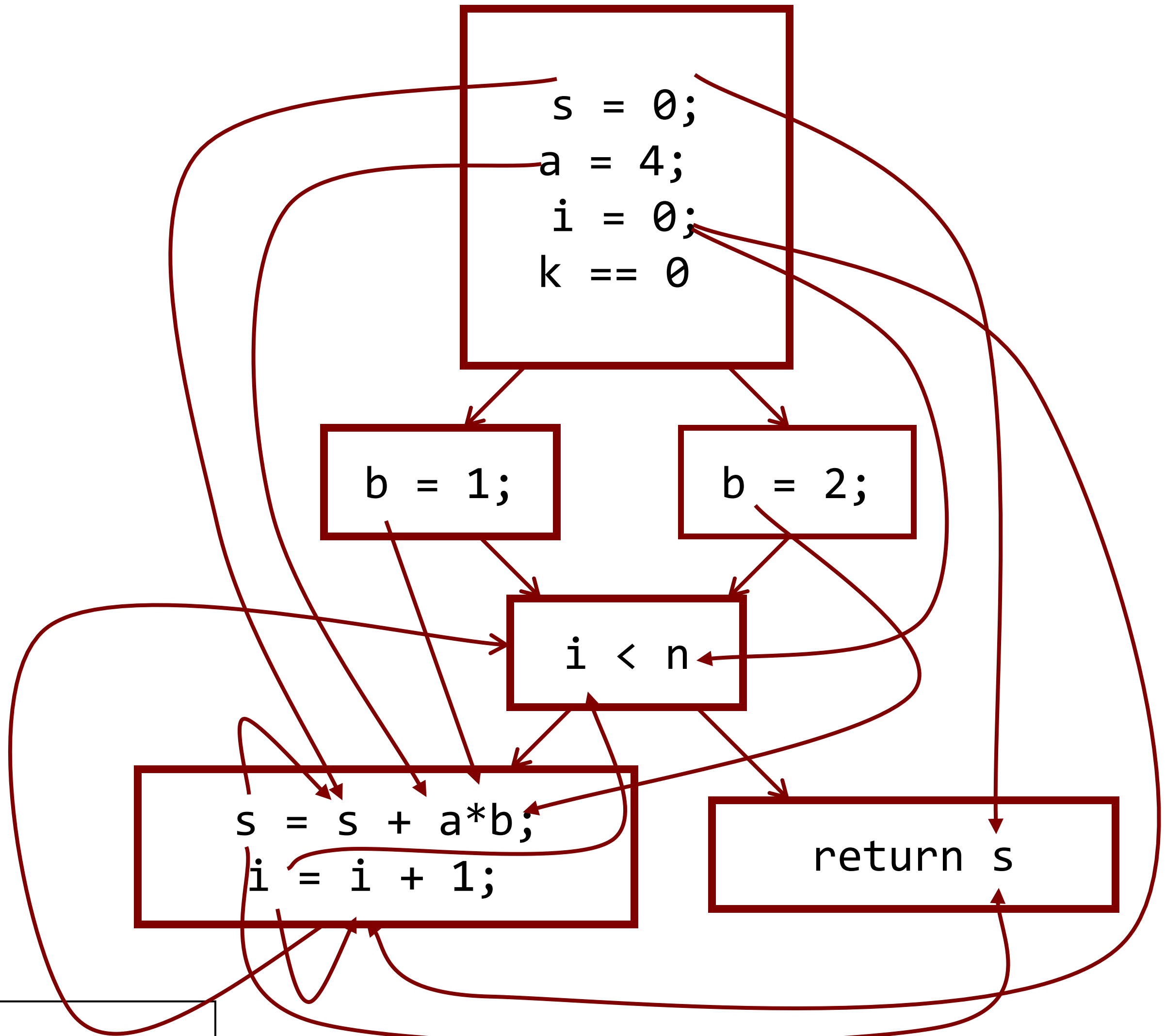
Concept of definition and use

- a = x+y
- is a definition of a
- is a use of x and y

A definition reaches a use if

- value written by definition
- may be read by use

Reaching Definitions



Example by
Saman Amarasinghe

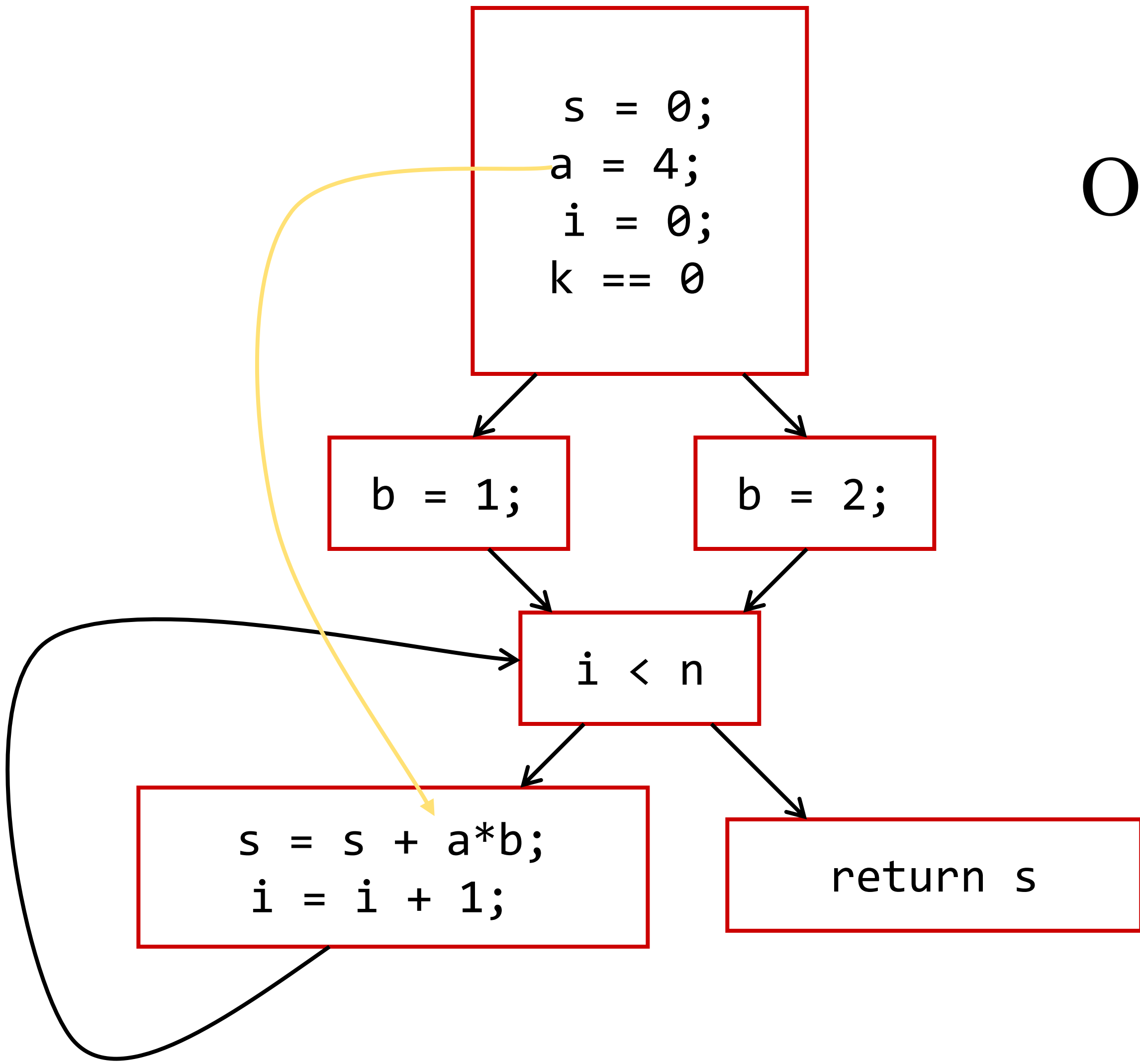
Reaching Definitions and Constant Propagation

Is a use of a variable a constant?

- Check all reaching definitions
- If all assign variable to same constant
- Then use is in fact a constant

Can replace variable with constant

Is a Constant in s = s+a*b?

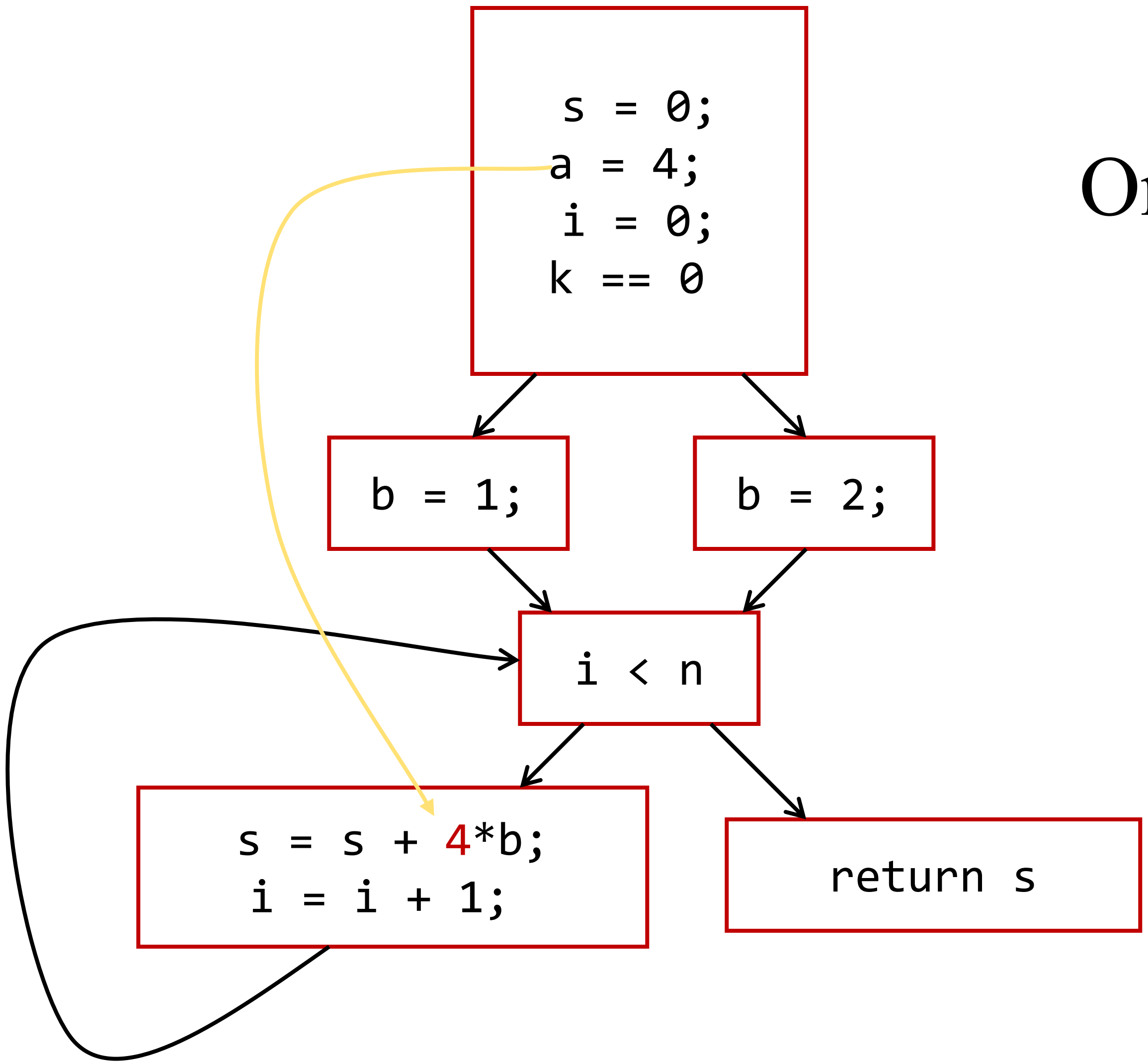


Yes!

On all reaching definitions

a = 4

Constant Propagation Transform

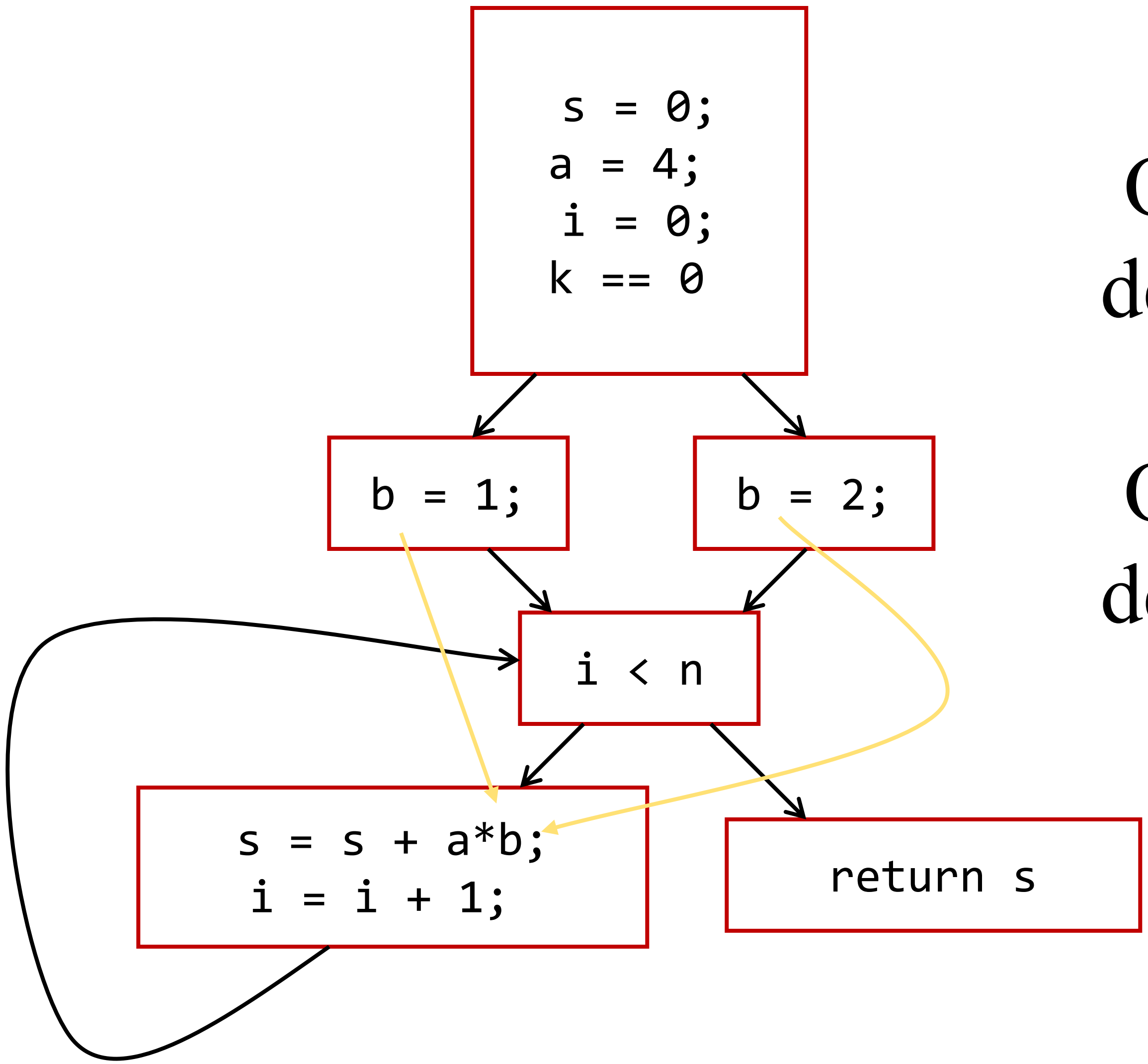


Yes!

On all reaching definitions

a = 4

Is b Constant in s = s+a*b?



No!

One reaching definition with

b = 1

One reaching definition with

b = 2

Computing Reaching Definitions

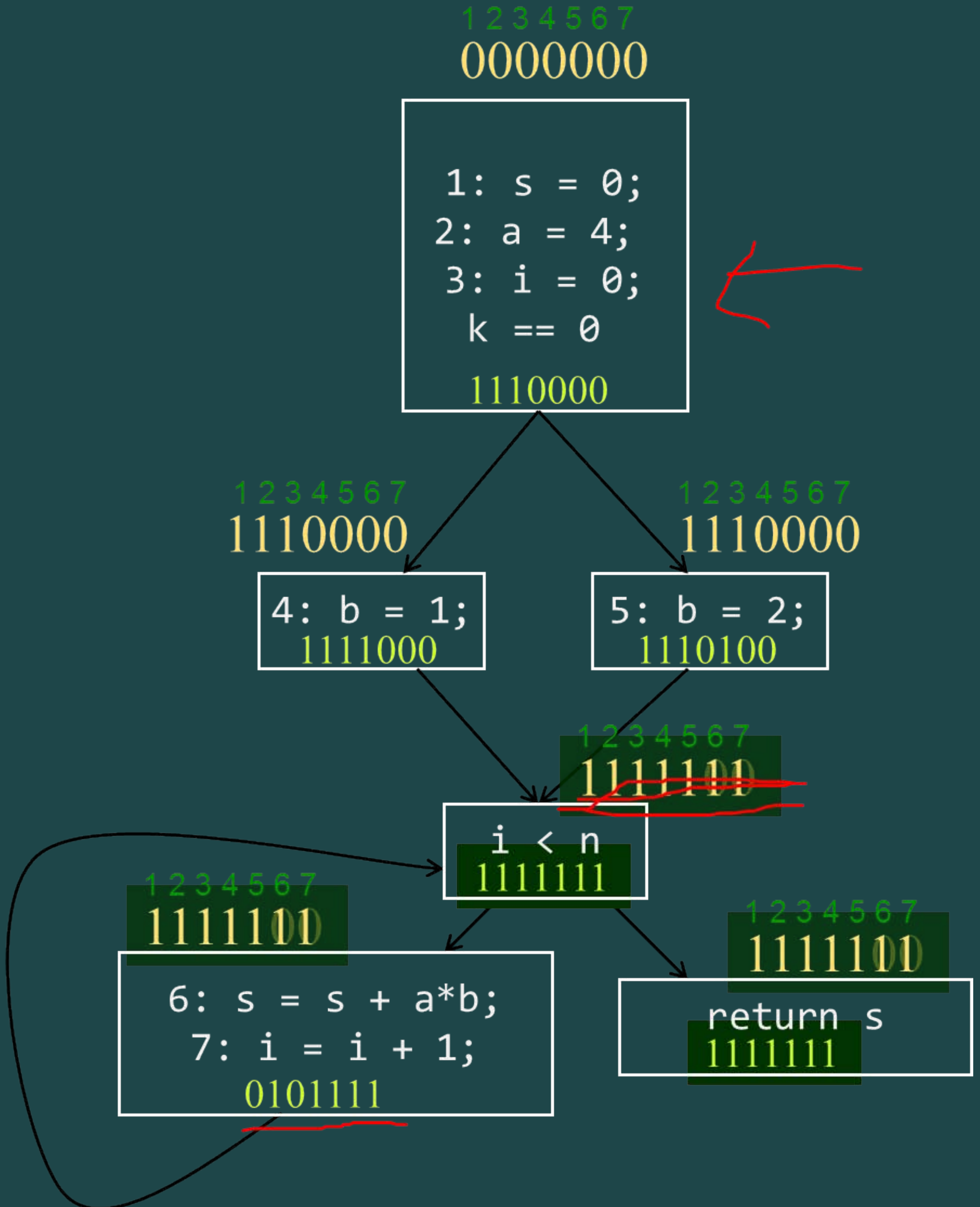
Compute with sets of definitions

- represent sets using bit vectors
- each definition has a position in bit vector

At each basic block, compute

- definitions that reach start of block
- definitions that reach end of block

Do computation by simulating execution of program until reach fixed point


1 2 3 4 5 6 7
0000000
1: s = 0;
2: a = 4;
3: i = 0;
k == 0
1110000
1 2 3 4 5 6 7
1110000
4: b = 1;
1111000
1 2 3 4 5 6 7
1110000
5: b = 2;
1110100
1 2 3 4 5 6 7
1111111
i < n
1111111
1 2 3 4 5 6 7
1111111
6: s = s + a*b;
7: i = i + 1;
0101111
1 2 3 4 5 6 7
1111111
return s
1111111

Transfer functions

Each basic block has

- IN - set of definitions that reach beginning of block
- OUT - set of definitions that reach end of block
- GEN - set of definitions generated in block
- KILL - set of definitions killed in block

GEN[s = s + a*b; i = i + 1;] = 0000011

KILL[s = s + a*b; i = i + 1;] = 1010000

Analyzer scans each basic block to derive GEN and KILL sets for each function

Dataflow Equations

IN[b] = OUT[b1] U ... U OUT[bn]

- where b1, ..., bn are predecessors of b in CFG

OUT[b] = (IN[b] - KILL[b]) U GEN[b]

IN[entry] = 0000000

Result: system of equations

Solving Equations

Use fixed point algorithm

Initialize with solution of OUT[b] = 0000000

Repeatedly apply equations

- IN[b] = OUT[b1] U ... U OUT[bn]
- OUT[b] = (IN[b] - KILL[b]) U GEN[b]

Until reach fixed point

Until equation application has no further effect

Use a worklist to track which equation applications may have a further effect

Questions

Does the algorithm halt?

- yes, because transfer function is monotonic
- if increase IN, increase OUT
- in limit, all bits are 1

If bit is 0, does the corresponding definition ever reach basic block?

If bit is 1, is does the corresponding definition always reach the basic block?

6.820 Fundamentals of Program Analysis
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