Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.685 Electric Machines

Problem Set 2 solutions — September 18, 2013

Problem 1: Torque is, for this system, $T^e = -Mi_1 i_2 \sin\theta$ and voltage in coil 1 is $v_1 = \frac{d\lambda_1}{dt} = \frac{d}{dt} M \cos\theta i_2$.

1. if $i_2 =$ constant, then the answers are trivial:

$$\begin{aligned} T^e &= MI_1 I_2 \sin\omega t \\ v_1 &= -\omega M I_2 \sin\omega t \end{aligned}$$

These are plotted in Figure 1

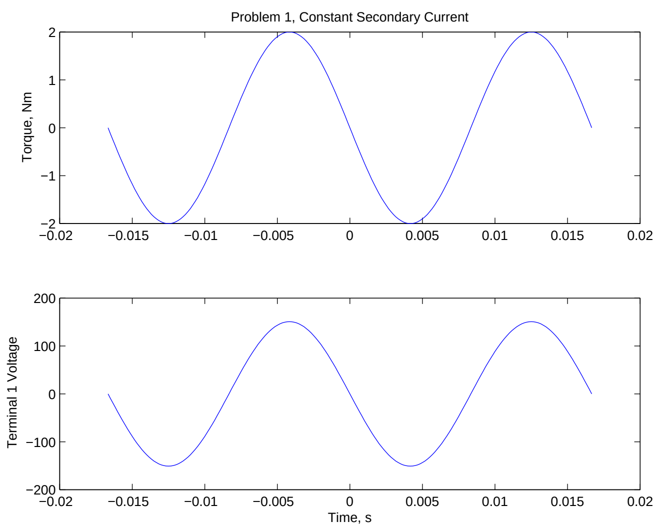

Figure 1: Problem 1: Fixed Current in Coil 2

2. if $\lambda_2 = 0$,

$$i_2 = -\frac{M}{L_2} I_1 \cos\omega t$$

Torque is straightforward:

$$T^e = \frac{(MI_1)^2}{L_2} \sin\omega t \cos\omega t = \frac{(MI_1)^2}{2L_2} \sin 2\omega t$$

Flux in coil 1 is:

$$\lambda_1 = -M \cos\omega t \frac{M}{L_2} I_1 \cos\omega t = -\frac{M^2 I_1}{L_2} \cos^2\omega t = -\frac{M^2 I_1}{2L_2}(1 + cos 2\omega t)$$

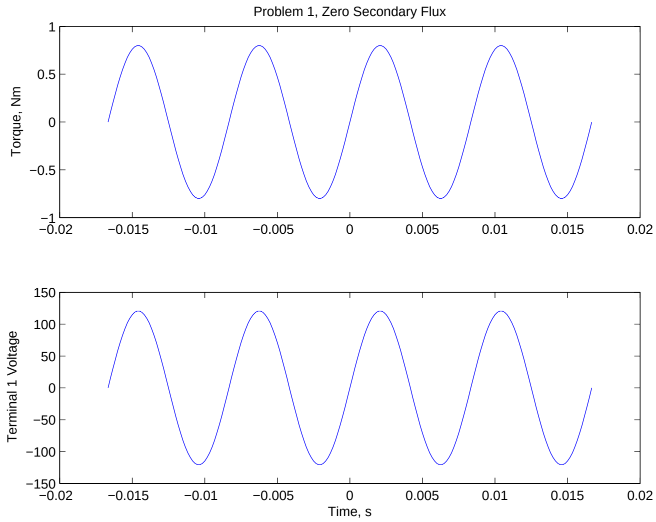

Figure 2: Problem 1: Zero flux in Coil 2

Differentiating this we have

$$v_1 = \frac{M^2 I_1}{L_2} \sin 2\omega t$$

These are plotted in Figure 2

3. Connected to a resistor, this is most conveniently worked in complex amplitudes:

$$V_2 = -RI_2 = j\omega L_2 I_2 + j\omega M I_2$$

or

$$I_2 = \frac{-j\omega M}{R + j\omega L_2} I_1$$

which gives us

$$i_2 = -|I_2| \cos(\omega t + \phi)$$

where

$$\begin{aligned} |I_2| &= \frac{\omega M}{\sqrt{R^2 + (\omega L_2)^2}} I_1 \\ \phi &= \frac{\pi}{2} - \tan^{-1} \frac{\omega L_2}{R} \end{aligned}$$

Torque is:

$$\begin{aligned} T^e &= -MI_1|I_2| sin\omega t \cos(\omega t + \phi) \\ &= -\frac{MI_1|I_2|}{2} (\sin(2\omega t + \phi) - \sin\phi) \end{aligned}$$

voltage in coil 1 is $v_1 = \frac{d\lambda_1}{dt}$ which can be evaluated to be:

$$\begin{aligned} v_1 &= -\omega M \sin \omega t i_2 + M \cos \omega t \frac{di_2}{dt} \\ &= \omega M |I_2| (\sin \omega t \cos(\omega t + \phi) + \cos \omega t \sin(\omega t + \phi)) \\ &= \omega M |I_2| \sin(2\omega t + \phi) \end{aligned}$$

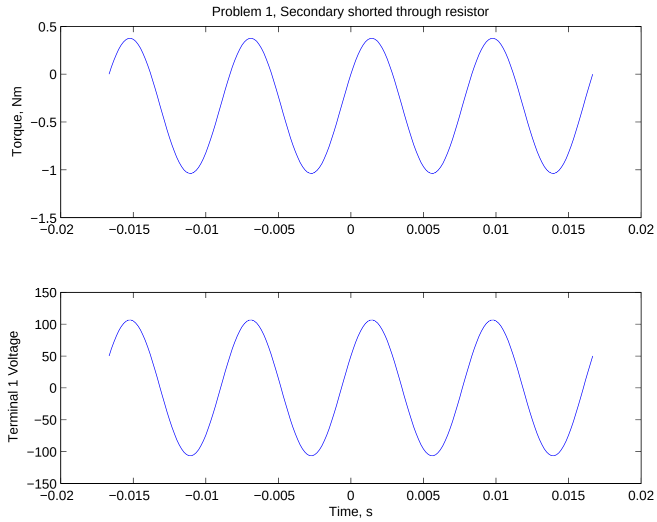

Figure 3: Problem 1: Coil 2 connected to 20 Ohm Resistor

These are plotted in Figure 3

4. The last part of this is simply taking the average value of torque and plotting over a range of speeds. This is plotted in Figure 4.

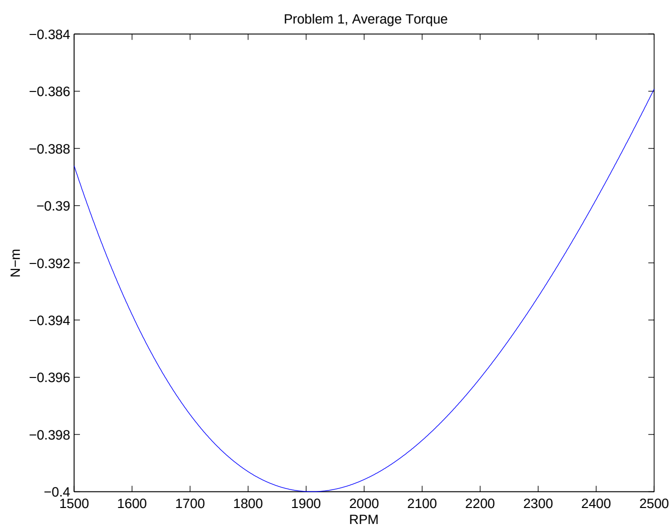

Figure 4: Problem 1: Average Torque

Problem 2: 1. using the principal of virtual work,

$$W'_m = \frac{1}{2}Li^2 + \lambda i$$

We don't know L but if it doesn't vary it doesn't matter, and force is just

$$f^e = \frac{\partial W'_m}{\partial x} = i\frac{\partial \lambda}{\partial x}$$

From the voltage 'test' described in the problem statement:

$$v = \frac{d\lambda}{dt} = \frac{d\lambda}{dx}\frac{dx}{dt}$$

and if $u = \frac{dx}{dt} = 5\text{m/s}$, and we can also use $t = \frac{x}{u}$, so that:

$$\frac{d\lambda}{dx} = \frac{A}{u^2}xe^{-\frac{\alpha}{u}x}$$

and then force is:

$$f^e = i\frac{d\lambda}{dx} = (40e)xe^{-100x}$$

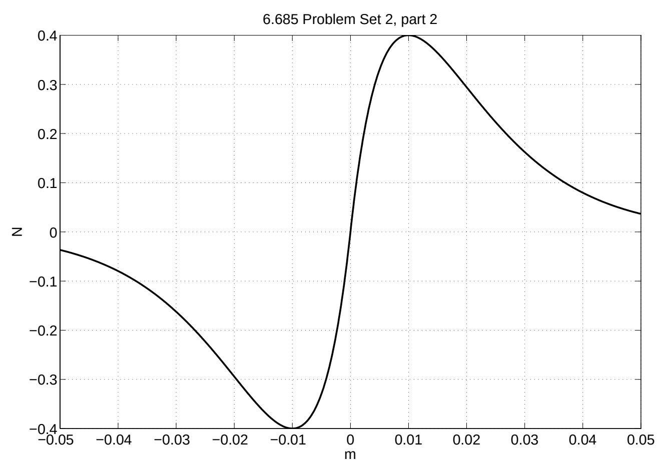

Figure 5: Force vs. position

2. Energy required to lift the magnet is $W = mgh = .05 * 9.812 * 1 = 0.49J$. This is also force integrated over distance:

$$W = \int_{0.01}^{\infty} I_0 40exe^{-100x}dx$$

Using

$$\int xe^{\alpha x}dx = \frac{1}{\alpha^2}(\alpha x - 1)e^{\alpha x}$$

where $\alpha = -100$,

$$\int_{.01}^{\infty} xe^{-100x}dx = 10^{-4}\left(100 \times .01 + 1\right)e^{-100\times .01} = \frac{2}{e} \times 10^{-4}$$

Then required current is:

$$I = \frac{0.49}{80 \times 10^{-4}} \approx 61.25A$$

Problem 3: 1. The rail width is such that, when the yoke is centered, each side has an overlap of $\frac{1}{2}w$.
Area is $\frac{1}{20}$ square meter, so force per unit area must be:

$$P = \frac{F}{A} = \frac{50,000}{.05} = 10^6 Pa = \frac{B^2}{2\mu_0}$$

Then flux density across the gap must be:

$$B = \sqrt{2\mu_0 \frac{F}{A}} \approx 1.58T$$

Required current is:

$$NI = 2g\frac{B}{\mu_0} \approx 37,720\text{A-T}$$

2. To accommodate lateral motion, note that total permeance of the system is:

$$\mathcal{P} = \frac{1}{\frac{1}{\mathcal{P}_1} + \frac{1}{\mathcal{P}_2}}$$

where

$$\mathcal{P}_1 = \frac{\mu_0 \ell \left(\frac{w}{2} - x\right)}{g}$$
$$\mathcal{P}_2 = \frac{\mu_0 \ell \left(\frac{w}{2} + x\right)}{g}$$

then

$$\mathcal{P} = \frac{\mu_0 \ell}{g} \frac{\left(\frac{w}{2}\right)^2 - x^2}{w}$$

Suspension Force is:

$$F = -\frac{(NI)^2}{2}\frac{\partial \mathcal{P}}{\partial g}$$

Or, inverting, required ampere-turns is:

$$NI = \sqrt{\frac{8Fwg^2}{\mu_0 \left(w^2 - 4x^2\right)}}$$

This is plotted in the top part of Figure 2.

3. Lateral force is found by differentiating the permeance with respect to lateral position x:

$$F_x = \frac{(NI)^2}{2}\frac{\partial \mathcal{P}}{\partial x} = -\frac{(NI)^2}{2}\frac{2x}{w}\frac{\mu_0 \ell}{g}$$

4. Since we have already calculated NI(x), we may combine that expression and the derivative of permeance with respect to x to get:

$$F_x = -\frac{8gFx}{w^2 - 4x^2}$$

This is plotted in the lower half of Figure 2.

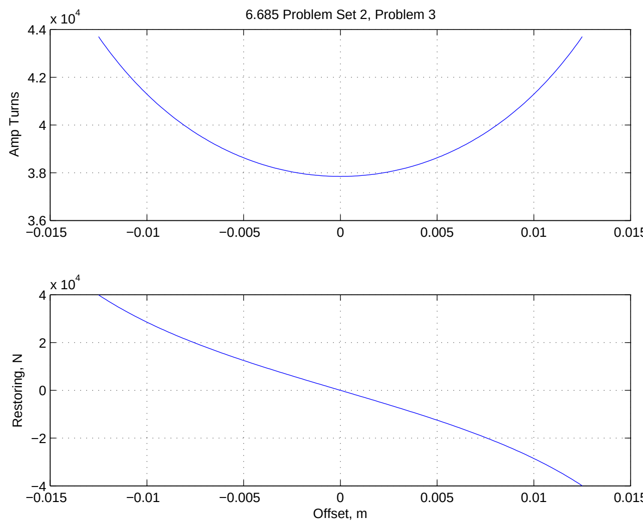

Figure 6: Required Ampere-Turns and Lateral Force

```
% 6.685 Problem Set 2, Problem 1
% Parameters
L1 = .1;
L2 = .1;
M = .08;
I1 = 5;
R = 20;
om = 2*pi*60;
t =-1/60:1/6000:1/60;

% Part 1: Constant Current
I2 = 5;
Tm1 = M*I1*I2;
Vm1 = om*M*I2;

T_e1 = -M*I1*I2 .* sin(om .* t);
V11 = -om*M*I2 .* sin(om .* t);

fprintf('Problem Set 2, Problem 1\n')
fprintf('Part 1: Torque Magnitude = %g\n', Tm1)
fprintf('Part 1: Voltage Magnitude = %g\n', Vm1)

figure(1)
subplot 211
plot(t, T_e1)
title('Problem 1, Constant Secondary Current')
ylabel('Torque, Nm')
subplot 212
plot(t, V11);
ylabel('Terminal 1 Voltage')
xlabel('Time, s')

% Part 2: Rotor shorted

Tm2 = (M*I1)^2/(2*L2);
Vm2 = om*M^2*I1/L2;

T_e2 = (M^2/(2*L2))*I1^2 .* sin(2*om .*t);
V12 = (om*M^2*I1/L2) .* sin(2*om .*t);
fprintf('Part 2: Torque Magnitude = %g\n', Tm2)
fprintf('Part 2: Voltage Magnitude = %g\n',Vm2)

figure(2)
subplot 211
plot(t, T_e2)
title('Problem 1, Zero Secondary Flux')
ylabel('Torque, Nm')
subplot 212
```

```
plot(t, V12);
ylabel('Terminal 1 Voltage')
xlabel('Time, s')

% Part 3: Rotor shorted through a resistor

I2m = om*M*I1/sqrt((om*L2)^2+R^2);
phi = atan(om*L2/R);
theta = pi/2 - phi;

T_e3 = -(M*I1*I2m/2) .* (sin(theta) - sin(2*om .* t + theta));
V13 = om*M*I2m .* sin(2*om .* t + theta);

figure(3)
subplot 211
plot(t, T_e3)
title('Problem 1, Secondary shorted through resistor')
ylabel('Torque, Nm')
subplot 212
plot(t, V13);
ylabel('Terminal 1 Voltage')
xlabel('Time, s')

% part 4: Average torque over a speed range

RPM = 1500:2500;

Om = (2*pi/60) .* RPM;
I_2 = I1 .* Om .* M ./ sqrt(R^2 + (Om .* L2) .^2);
Theta = pi/2 - atan(Om .* L2 ./ R);

Tav = -.5*M*I1 .* I_2 .* sin(Theta);

figure(4)
plot(RPM, Tav)
title('Problem 1, Average Torque');
ylabel('N-m')
xlabel('RPM')
```

```
% 6.685 Problem set 2, problem 2, 2013
a = 500;
A = 1000*exp(1);
%x0 = .01:.01:1;
t = -.015:.0001:.015;
f = A .* t .* exp(-a .* abs(t));
%g = exp(-a .* abs(x)) .* (-a .* abs(x) -1) ./ a^2;

figure(1)
%subplot(211)
plot(t, f)
%subplot(212)
%plot(x, g)
title('Coil Voltage')
xlabel('time, mS')
ylabel('Volts')
axis([-.015 .015 -2.5 2.5])
grid on

% now we can plot force vs. position
u = 5;
x = -.15:.001:.15;
D = (A/u^2) .* x .* exp(-(a/u) .* abs(x));
figure(2)
plot(x, D)
title('d flux/dx')
ylabel('Wb/m')
xlabel('distance from center, m')
axis([-.15 .15 -.5 .5])
grid on

% now get energy stored
B = A/u^2;
b = a/u;
x = 0:.0001:.15;
W_0 = -(B/b) .* x .* exp(-b .* x) + (B/b^2) .*  (1-exp(-b .* x));           % energy with respe
figure(4)
plot(x, W_0)
title('Energy from center of coil')
ylabel('Joules')
xlabel('Distance from coil center, m')

x0 = .01;                                   % to get energy with respect to this position
xx = .01:.0001:.15;
W = (B/b) .* (x0 * exp(-b * x0) - xx  .* exp(-b .* xx)) + (B/b^2) .* (exp(-b*x0) - exp(-b .* xx
M = .05;
v = sqrt((2/M) .* W);
```

```
x1 = .1;
W1 = (B/b) *(x0 *exp(-b*x0) - x1*exp(-b*x1)) + (B/b^2) * (exp(-b*x0) - exp(-b*x1));
v0 = sqrt((2/M) * W1);

fprintf('Velocity = %g\n', v0)

figure(3)
plot(xx, v)
title('Velocity vs. x')
ylabel('m/s')
xlabel('position, m')
```

```
% 6.685, Fall, 2013 problem set 2, problem 3

F = 50000;       % required force
muzero = pi*4e-7;

W = .1;                    % width
l = 1;                     % length
g = .015;                  % gap
A = W*l;                   % resulting area
B = sqrt(2*muzero*F/A);
NIz = 2*g*B/muzero;
fprintf('Required Supporting Flux Density %g T\n', B)
fprintf('Required Coil Current            %g A-t\n',NIz)

d=W/400;                   % increment of x
x = -W/4:d:W/4;            % over this range of x
NIsq = 8*F*W*g^2 ./(muzero*l*(W^2- 4 .*x .^2));
NI =sqrt(NIsq);
Fx = - 8*g*F .*x ./(W^2 - 4 .*x .^2);

figure(1)
subplot 211
plot(x, NI)
title('6.685 Problem Set 2, Problem 3')
ylabel('Amp Turns')
grid on
subplot 212
plot(x, Fx)
ylabel('Restoring, N')
xlabel('Offset, m')
grid on
```