MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

Receivers, Antennas, and Signals – 6.661

Problem Set No. 13 *and solutions*   Issued: 5/6/03   Due: *Never*

Note: *Quiz 2 is scheduled for May 15 during class, and will begin promptly at 2:30 PM.*
The quiz is open book, and a calculator may be used (it probably will be useful).
The quiz will emphasize all material covered after March 25, i.e., all material not
covered on the first quiz.

Problem 13.1

A gas inside a hollow blackbody has kinetic temperature T_k. Two of the energy
levels, E_1 and E_2, in the gas molecules of interest have relative populations of n_1 and n_2,
respectively. The blackbody has temperature T_{bb}, which produces a radiation field of
brightness temperature T_{bb}.

a) Consider the limit where the intermolecular collisions of the gas are very frequent (N
sec^{-1}) compared to the nominal lifetime τ of the state (τ(sec) $\cong$ 1/A; A is the "Einstein
A"). What is the effective temperature T_{eff} characterizing the relative values of n_1, n_2,
E_1, and E_2 for this special case where N >> A?

b) Repeat (a) for the case where intermolecular gas collisions do not occur (N→0).

c) What are the relative values of n_1 and n_2 for case (a)?

Problem 13.2

A communications system transmits packets of one-millisecond duration and
containing 1000 bits. The packets are transmitted through the ionosphere with group
velocity v_g. If the plasma frequency is 20 MHz and the signal frequency is 30 MHz, how
long a transmission path L[m] can we have before the distortion of the bit stream due to
ionospheric dispersion becomes severe?

Problem 13.3

A certain measurement d of parameter p has the form d = a + p + n, where a is a
constant and n is white noise uniformly distributed between ±a.

a) What is the least-square-error estimate for p if d = a? [Hint: a figure might help here.]

b) Repeat (a) for the case where d = 2a.

c) What is the maximum-aposteriori (MAP) estimate for p for case (b), if any? Explain.

Problem 13.4

What would be the shapes of the temperature weighting functions W(f,z) appropriate to measuring temperature profiles inside deep layers of molten glass using multispectral measures of thermal radiation?

Problem 13.5

How plausible is it that a good *linear* estimator for the personal income of single people could be based on 100 numerical credit-card-based measures, including their telephone bills, number of foreign phone calls, numbers of shoes purchased, restaurant charges, etc., given that the relations between these parameters and income vary and are non-linear? (E.g., no matter how wealthy someone is, his or her time on the phone is limited.) Assume the personal incomes and annual credit card records for 1000 random single people are available to help develop the estimator.

a) Explain your answer.

b) What sorts of non-linear estimators might be useful for this problem?   Some imagination might help here.

Solutions

Problem 13.1

a) *When collisions are more frequent than other events influencing level populations, the level populations are determined by the kinetic temperature of the gas,* $\boxed{T_{eff} = T_k.}$

b) *When there are no collisions, transitions between levels are dominated by radiative processes, so* $\boxed{T_{eff} = T_{bb},}$ *the radiation temperature determined by the surrounding blackbody.*

c) *The Boltzmann distribution for the average number of atoms in a given state is:*
   $\bar{n} \propto e^{-E/kT}$, *so* $\boxed{n_1/n_2 \cong e^{(E_2 - E_1)/kT}.}$

Problem 13.2

Distortion occurs when the high-frequency components of the signal stream arrive at a time offset from that for the low-frequency components by more than the width of any symbol conveying information. In this case, each bit conveys information, and each lasts Δt = one microsecond ([0.001sec]/[1000bits/packet]). The frequency spread associated with this bit is $\Delta t^{-1} \cong 1$ MHz or more. In this plasma $v_g = c(1 - [\omega_p/\omega]^2)^{0.5} = c(1 - [4/9])^{0.5} = 0.75c$ ms^{-1}.   At a frequency 1 MHz higher the velocity is 0.019c greater.   These two velocities will spread the pulse one microsecond at the maximum distortion-free path L_{max}. Arrival time = L/v_g. $L_{max}(1/v_{min} - 1/v_{max}) = 10^{-6}$, so $L_{max} \cong 10^{-6}(1/v_{min} - 1/v_{max})^{-1} =$

$10^{-6}(1/0.75c - 1/0.769c)^{-1} = \boxed{9.1 \text{ km.}}$ *The ionosphere is much thicker than this, so we prefer to use frequencies > 2 GHz for satellite communications, and 10 GHz is better.*

Problem 13.3

a) *The probability distribution $p(p/a)$ is uniform over the interval $p = 0 \rightarrow 1$, and the least-square-estimate is thus $\boxed{a/2,}$ and not a.*

b) *If $d = 2a$, the same logic (see figure) suggests the estimate $\boxed{p = a.}$*

c) *The aposteriori probability distribution here is flat, so the maximum could be $\boxed{\text{anywhere from } p = 0 \text{ to } p = 2a.}$*
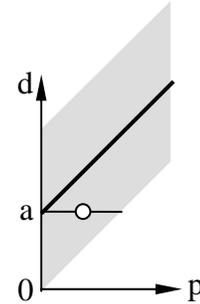
Problem 13.4

Let's assume the absorption coefficient $\alpha(f,z)$ of glass is independent of temperature so that $\alpha(f,z) = \alpha(f)$, and the slab is so thick that it becomes opaque. Then $T_B = \int_0^z T(z)\alpha(f,z)e^{-\tau(z)}dz$ where $\tau(z) = \int_0^z \alpha(f,z)dz \cong z\alpha(f)$. Then $\boxed{W(z,f) = \alpha(f,z)e^{-\tau(z)} \cong \alpha(f)e^{-z\alpha(f)}.}$ Thus the weighting functions in this case would be decaying exponentials with depth. If the absorption coefficient were temperature dependent, then the weighting functions would become temperature dependent too.

Problem 13.5

a) *Most credit-card based measures within purchase categories will vary non-linearly with the personal income of the card holder. For example, there will be essentially zero purchases of items costing more than one's annual income, or some reasonable fraction thereof. And, no matter how wealthy they are, essentially no one will purchase more of something than they can consume in a reasonable time, for example, toothpaste. These non-linearities are plausibly different for different categories of expenditure, no matter what they are. Because there are 100 such non-linear measures, it is plausible that many possible combinations would yield, on average, an estimate linearly related to personal income, even if the variance of the estimate were high. Averaging over 1000 people to determine the influence of 100 metrics should be reasonably robust.*

b) *Other non-linear credit-card metrics can also be imagined, such as payment history, variances, and fluctuation behavior. In general, metrics relating to the fraction of expenditures for luxury items relative to expenditures for basic needs would be helpful, where ratios are non-linear. Also, the presence of clusters of high-cost purchases versus rationed expenditures could also be informative, particularly if they are discretionary, implying the bank account from which the card bill is paid must be large.*