6.642 Continuum Electromechanics
Fall 2008

6.642, Continuum Electromechanics

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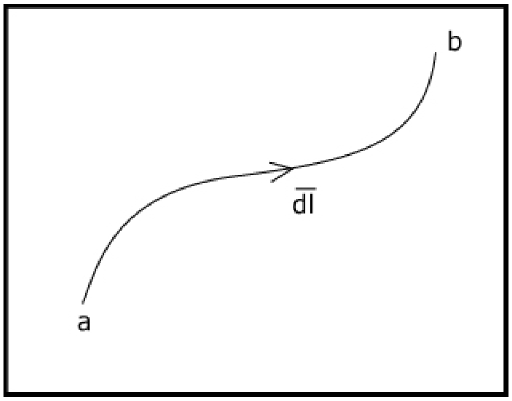
Lecture 5: Laws, Approximations, and Relations of Fluid Mechanics

Continuum Electromechanics (Melcher) – Sections 7.1-7.8

I. Useful Vector Operations and Identities

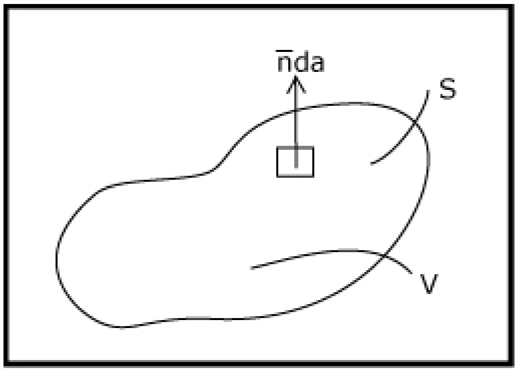
Gradient

$$\int_a^b \nabla \chi \bullet \overline{dl} = \chi(b) - \chi(a)$$



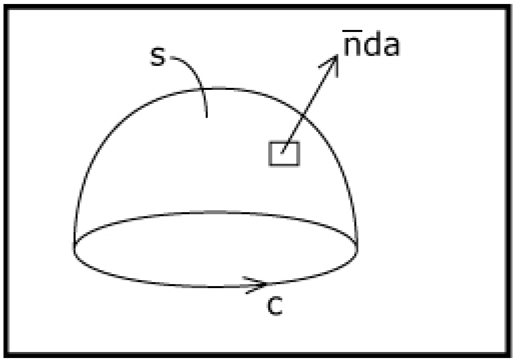
Gauss's Law (Divergence Theorem)

$$\int_V \nabla \bullet \overline{A}\, dV = \oint_S \overline{A} \bullet \overline{n}\, da$$



Stokes' Theorem

$$\int_S \nabla \times \overline{A} \bullet \overline{n}\, da = \oint_C \overline{A} \bullet \overline{dl}$$



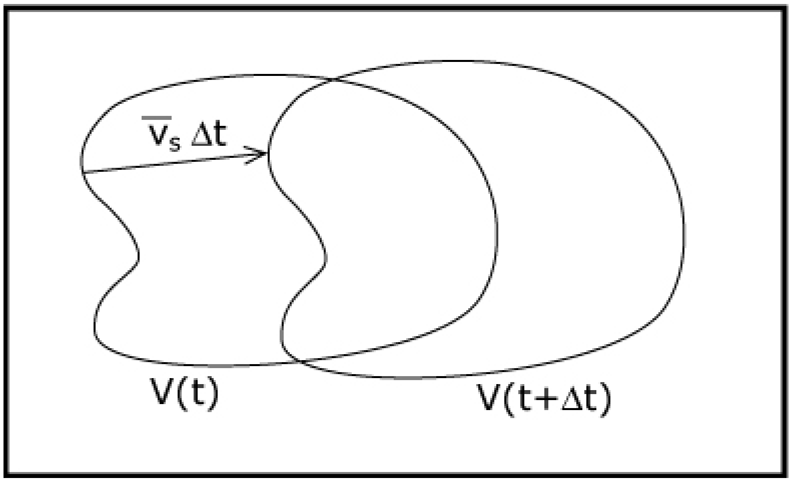
Some useful Vector Identities

$$\nabla \times (\nabla f) = 0$$

$$\nabla \bullet \left(\nabla \times \overline{A}\right) = 0$$

$$\left(\overline{A} \times \overline{B}\right) \bullet \overline{C} = \overline{A} \bullet \left(\overline{B} \times \overline{C}\right)$$ (Dot and Cross can be interchanged in the scalar triple product)

II. Time Derivative of a Fluid Volume Integral



ζ = any scalar quantity such as density ρ

$$\frac{d}{dt}\int_{V(t)} \zeta(t)\, dV = \lim_{\Delta t \to 0} \frac{\int_{V(t+\Delta t)} \zeta(t+\Delta t)\, dV - \int_{V(t)} \zeta(t)\, dV}{\Delta t}$$

Linearize all terms to first order in Δt

$$\zeta(t+\Delta t) = \zeta(t) + \frac{\partial \zeta}{\partial t}\Delta t + ...$$

$$\frac{d}{dt}\int_{V(t)} \zeta(t)\, dV = \lim_{\Delta t \to 0} \frac{\int_{V(t+\Delta t)} \zeta(t)\, dV - \int_{V(t)} \zeta(t)\, dV + \int_{V(t+\Delta t)} \frac{\partial \zeta}{\partial t}\Delta t\, dV}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\int_{\Delta V} \zeta(t)\, dV + \int_{V(t)} \frac{\partial \zeta}{\partial t}\Delta t\, dV}{\Delta t}$$

$$\Delta V = \overline{v}_s \Delta t \bullet \overline{da} \quad \left(\overline{v}_s = \text{fluid surface velocity}\right)$$

$$\frac{d}{dt}\int_{V(t)} \zeta(t)\, dV = \lim_{\Delta t \to 0} \frac{\oint_S \zeta(t)\overline{v}_s \bullet \cancel{\Delta} t\, \overline{da} + \int_{V(t)} \frac{\partial \zeta}{\partial t}\cancel{\Delta} t\, dV}{\cancel{\Delta} t}$$

$$\frac{d}{dt}\int_{V(t)} \zeta(t)\, dV = \int_V \frac{\partial \zeta}{\partial t}\, dV + \oint_S \zeta \overline{v}_s \bullet \overline{da}$$

$$\oint_S \zeta \bar{v}_s \cdot \overline{da} = \int_V \nabla \cdot \left(\zeta \bar{v}\right) dV \qquad \text{(Divergence Theorem)}$$

$$\frac{d}{dt} \int_{V(t)} \zeta(t)\, dV = \int_V \left[\frac{\partial \zeta}{\partial t} + \nabla \cdot \left(\zeta \bar{v}\right)\right] dV$$

III. Conservation of Mass $(\rho = \zeta)$

$$\frac{d}{dt} \int_V \rho\, dV = 0 = \int_V \frac{\partial \rho}{\partial t} dV + \underbrace{\oint_S \rho \bar{v}_s \cdot \bar{n}\, da}_{\int_V \nabla \cdot \left(\rho \bar{v}\right) dV}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \bar{v}\right) = 0 \text{ (Volume is arbitrary)}$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \left(\bar{v} \cdot \nabla\right) \rho}_{} + \rho \nabla \cdot \bar{v} = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \bar{v} = 0$$

Incompressible $\Rightarrow \dfrac{D\rho}{Dt} = 0 \Rightarrow \nabla \cdot \bar{v} = 0$

IV. Conservation of Momentum $(\rho v_i = \zeta)$, i^{th} component where i=x, y, or z

$$\frac{d}{dt} \int_V \rho v_i\, dV = \int_V F_i\, dV = \int_V \frac{\partial}{\partial t}\left(\rho v_i\right) dV + \oint_S \rho v_i \bar{v}_s \cdot \bar{n}\, da$$

$$\int_V \left[\frac{\partial}{\partial t}\left(\rho v_i\right) + \nabla \cdot \left(\rho v_i \bar{v}\right)\right] dV = \int_V F_i\, dV$$

$$\frac{\partial}{\partial t}\left(\rho v_i\right) + \nabla \cdot \left(\rho v_i \bar{v}\right) = F_i$$

$$\rho \left[\frac{\partial v_i}{\partial t} + \left(\bar{v} \cdot \nabla\right) v_i\right] + v_i \left[\underbrace{\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \bar{v}\right)}_{0 \text{ (Conservation of mass)}}\right] = F_i$$

$$\rho \left[\frac{\partial v_i}{\partial t} + \left(\bar{v} \cdot \nabla\right) v_i\right] = F_i$$

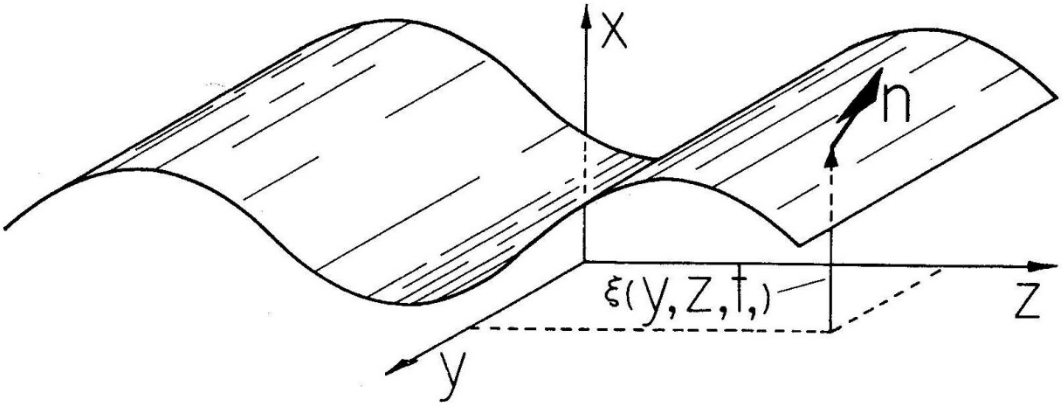
$$\rho \left[\frac{\partial \bar{v}}{\partial t} + \left(\bar{v} \cdot \nabla\right) \bar{v}\right] = \bar{F}$$

V. Equations of Motion for an Inviscid Fluid

$$\rho\left[\frac{\partial \bar{v}}{\partial t}+\left(\bar{v}\cdot\nabla\right)\bar{v}\right]=-\nabla p+\bar{F}_{ex}$$

VI. Eulerian Description of the Fluid Interface

$$F(x, y, z, t)=0=\xi(y, z, t)-x$$



Fluid interface.

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$$\bar{n}=\frac{\nabla F}{\left|\nabla F\right|}$$

$$\frac{DF}{Dt}=\frac{\partial F}{\partial t}+\left(\bar{v}\cdot\nabla\right)\bar{F}=0 \text{ on } F=0=\xi(y, z, t)-x$$

$$\frac{\partial \xi}{\partial t}+v_x\underbrace{\frac{\partial F}{\partial x}}_{-1}+v_y\underbrace{\frac{\partial F}{\partial y}}_{\frac{\partial \xi}{\partial y}}+v_z\underbrace{\frac{\partial F}{\partial z}}_{\frac{\partial \xi}{\partial z}}=0$$

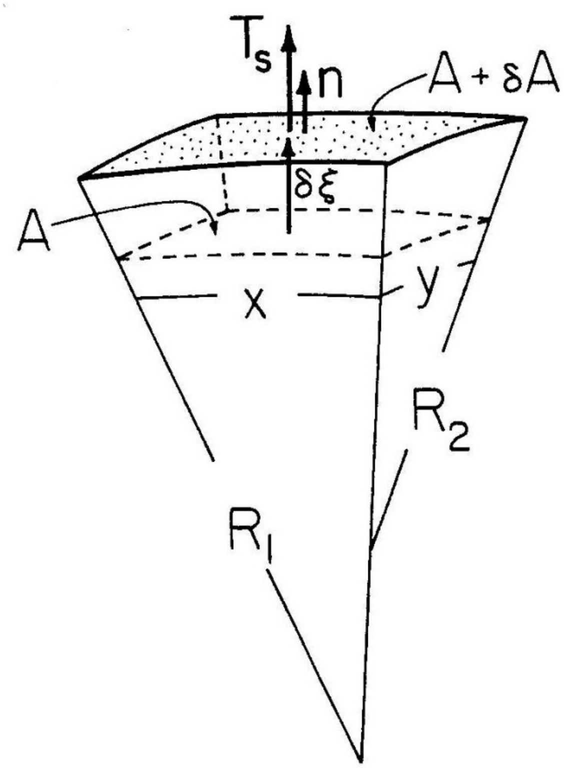
$$v_x=\frac{\partial \xi}{\partial t}+v_y\frac{\partial \xi}{\partial y}+v_z\frac{\partial \xi}{\partial z}$$

VII. Surface Tension

A. Surface Force Density

$$\delta W_s = \gamma\, \partial A$$

$$\underbrace{\delta W_s}_{\substack{\text{Increase}\\ \text{of surface}\\ \text{energy}}} + \underbrace{T_s A\, \delta\xi}_{\substack{\text{Work done on}\\ \text{interface}}} = 0 \Rightarrow \gamma\, \partial A + T_s A\, \delta\xi = 0$$



Section of interface that suffers perpendicular displacement δξ to make new surface δA.

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$$A + \delta A = (x + \delta x)(y + \delta y) \approx xy + y\delta x + x\delta y$$

$$\frac{x + \delta x}{R_1 + \delta\xi} = \frac{x}{R_1} \Rightarrow \delta x = \frac{x}{R_1}\delta\xi$$

$$\frac{y + \delta y}{R_2 + \delta\xi} = \frac{y}{R_2} \Rightarrow \delta y = \frac{y}{R_2}\delta\xi$$

$$\delta A = y\delta x + x\delta y$$

$$= \left(\frac{xy}{R_1} + \frac{xy}{R_2}\right)\delta\xi$$

$$= xy\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\delta\xi$$

$$= A\ \delta\xi\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

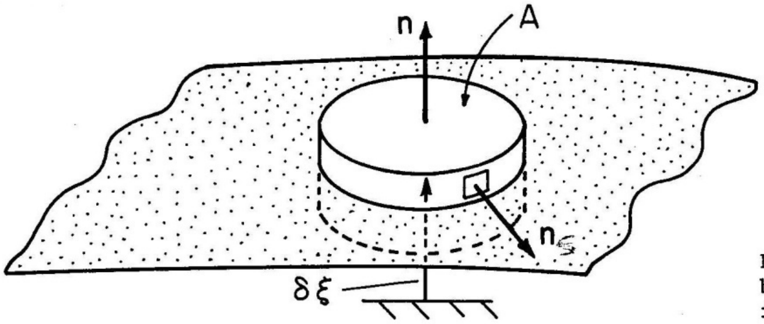
$$\gamma A\ \delta\xi\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + T_s A\ \delta\xi = 0$$

$$\overline{T}_s = -\gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\overline{n}$$ [Young and Laplace surface force density]

B. Interfacial Deformation

$$\int_V \nabla \bullet \overline{C}\, dV = \oint_S \overline{C} \bullet \overline{n}_s\, da$$ (Divergent Theorem)

$$\overline{C} = \overline{n}$$



Elemental volume V enclosed by surface S intersecting interface between fluids.

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$\overline{n} = \overline{n}_s$ on top

$\overline{n} = -\overline{n}_s$ on bottom

$\overline{n} \perp \overline{n}_s$ on side

$$\int_V \nabla \bullet \overline{C}\, dV \approx \nabla \bullet \overline{n}\, \delta\xi A = \oint_S \overline{n} \bullet \overline{n}_s\, dA$$

$$= A_{top} - A_{bottom} = \delta A$$

$$\left(\nabla \bullet \overline{n}\right) A\delta\xi = \delta A$$

$$T_s = \frac{-\gamma\delta A}{A\delta\xi} = \frac{-\gamma \cancel{A\delta\xi}\, \nabla \cdot \bar{n}}{\cancel{A\delta\xi}}$$

$$= -\gamma \nabla \cdot \bar{n} \Rightarrow \nabla \cdot \bar{n} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\bar{T}_s = -\gamma\left(\nabla \cdot \bar{n}\right)\bar{n}$$

VIII. Boundary Conditions

A. Rigid Wall

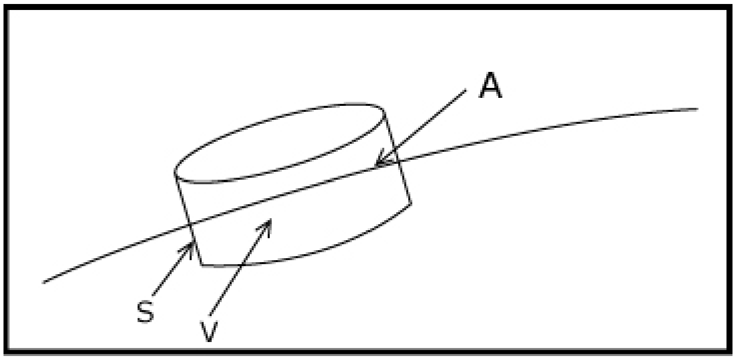
$\bar{n} \cdot \bar{v} = 0$ (normal velocity component is zero)

$\bar{n} \times \bar{v} = 0$ (Viscous flow) (tangential velocity component is zero)

B. Interface

$\bar{n} \cdot \| \bar{v} \| = 0$; $\| \bar{v} \| = \bar{v}_{above} - \bar{v}_{below}$

Force Equilibrium



$$\int_V F_i \, dV + \int_A \left(T_s\right)_i \, da = 0$$

$$\bar{F} = \bar{F}^m + \bar{F}^e$$

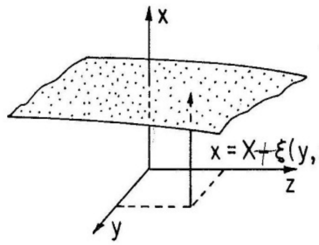
mechanical electrical

$$F_i^e = \frac{\partial T_{ij}^e}{\partial x_j}$$

$$\bar{F}^m = -\nabla p = \nabla \cdot \bar{\bar{T}}^m \Rightarrow \frac{\partial T_{ij}^m}{\partial x_j} = -\frac{\partial p}{\partial x_i} = -\delta_{ij}\frac{\partial p}{\partial x_j}$$

$$T_{ij}^m = -p\,\delta_{ij}$$

Summary of normal vector and surface tension surface force density for small perturbations from planar, circular cylindrical and spherical equilibria.

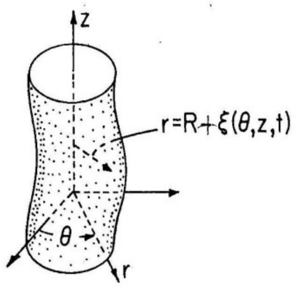


$$\vec{n} = \vec{i}_x - \frac{\partial\xi}{\partial y}\vec{i}_y - \frac{\partial\xi}{\partial z}\vec{i}_z \tag{a}$$

$$(\vec{T}_s)_x = \gamma\left(\frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\right) \tag{b}$$

$$\xi = \mathrm{Re}\,\tilde{\xi}\,\exp -j(k_y y + k_z z) \tag{c}$$

$$\tilde{T}_s = -\gamma(k_y^2 + k_z^2)\tilde{\xi} \tag{d}$$

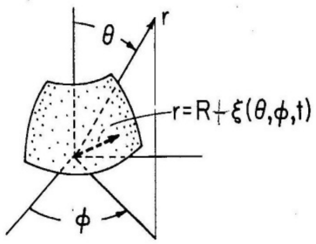


$$\vec{n} = \vec{i}_r - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\vec{i}_\theta - \frac{\partial\xi}{\partial z}\vec{i}_z \tag{e}$$

$$(\vec{T}_s)_r = \gamma\left[-\frac{1}{R} + \frac{\xi}{R^2} + \frac{1}{R^2}\frac{\partial^2\xi}{\partial\theta^2} + \frac{\partial^2\xi}{\partial z^2}\right] \tag{f}$$

$$\xi = \mathrm{Re}\,\tilde{\xi}\,\exp -j(m\theta + kz) \tag{g}$$

$$\tilde{T}_s = \frac{\gamma}{R^2}\left[(1 - m^2) - (kR)^2\right]\tilde{\xi} \tag{h}$$



$$\vec{n} = \vec{i}_r - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\vec{i}_\theta - \frac{1}{R\sin\theta}\frac{\partial\xi}{\partial\phi}\vec{i}_\phi \tag{i}$$

$$(\vec{T}_s)_r = \gamma\left[-\frac{2}{R} + \frac{2\xi}{R^2} + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\xi}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2}\right] \tag{j}$$

$$\xi = \mathrm{Re}\,\tilde{\xi}\,P_n^m(\cos\theta)e^{-jm\phi} \tag{k}$$

$$\tilde{T}_{sr} = -\frac{\gamma}{R^2}(n - 1)(n + 2)\tilde{\xi} \tag{ℓ}$$

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$$\int_V F_i\,dV = \oint_S \left(T_{ij}^m + T_{ij}^e\right)\left(\bar{i}_n\right)_j da$$

$$\left\|T_{ij}^m + T_{ij}^e\right\| n_j + \left(T_s\right)_i = 0$$

$$\|p\| n_i = \left\|T_{ij}^e\right\| n_j - \gamma\nabla\bullet\bar{n}\, n_i$$

IX. Bernoulli's Law

$$\overline{F}^{g} = \rho \overline{g} = \nabla\left(\rho \overline{g} \cdot \overline{r}\right) \qquad \text{if } \rho = \text{constant}, \quad \overline{r} = X\overline{i}_x + y\overline{i}_y = z\overline{i}_z$$

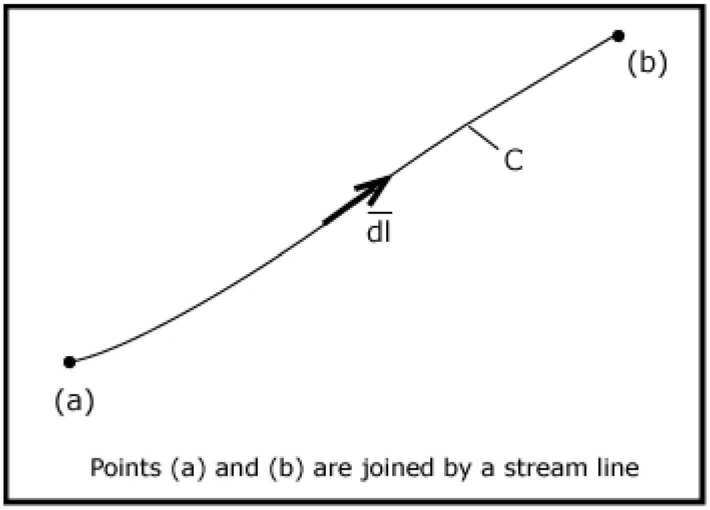
$\overline{F}^{e} = -\nabla \mathcal{E}$ [Special case when electrical force is written as gradient of scalar]

$$\rho\left[\frac{\partial \overline{v}}{\partial t} + \left(\overline{v} \cdot \nabla\right)\overline{v}\right] + \nabla p = \overline{F}^{ex} = \nabla\left(\rho\overline{g} \cdot \overline{r} - \mathcal{E}\right)$$

$$\left(\overline{v} \cdot \nabla\right)\overline{v} = \left(\nabla \times \overline{v}\right) \times \overline{v} + \frac{1}{2}\nabla\left(\overline{v} \cdot \overline{v}\right)$$

$$\rho\left[\frac{\partial \overline{v}}{\partial t} + \overline{\omega} \times \overline{v}\right] + \nabla\left[p + \frac{1}{2}\rho\,\overline{v} \cdot \overline{v} - \rho\overline{g} \cdot \overline{r} + \mathcal{E}\right] = 0$$

$\overline{\omega} = \nabla \times \overline{v}$ (vorticity)



Points (a) and (b) are joined by a stream line

$\overline{dl} \parallel \overline{v}$

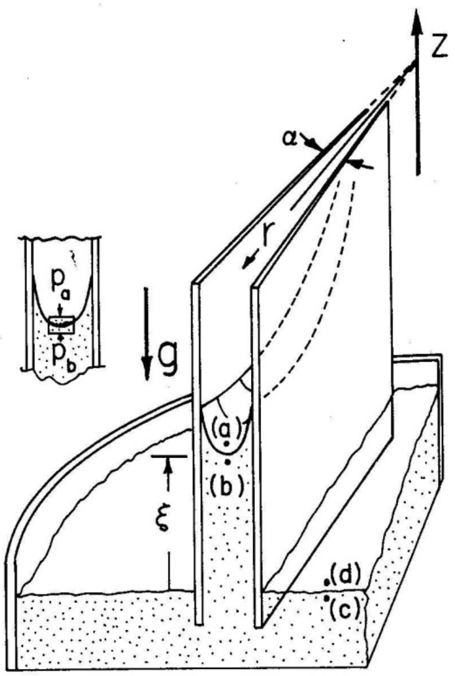
$$\rho\int_a^b \frac{\partial \overline{v}}{\partial t} \cdot \overline{dl} + \left[p + \frac{1}{2}\rho\,\overline{v} \cdot \overline{v} - \rho\overline{g} \cdot \overline{r} + \mathcal{E}\right]\Bigg|_a^b = 0$$

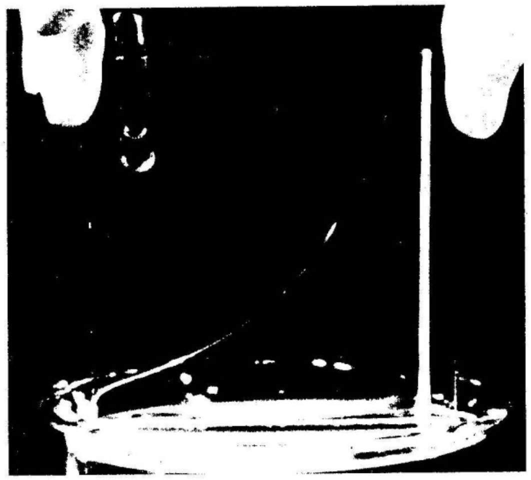
Irrotational Flows $\left(\nabla \times \overline{v} = \overline{\omega} = 0\right) \Rightarrow \overline{v} = -\nabla\theta$

$$\left[-\rho\frac{\partial \theta}{\partial t} + p + \frac{1}{2}\rho\,\overline{v} \cdot \overline{v} - \rho\overline{g} \cdot \overline{r} + \mathcal{E}\right]_a^b = 0$$

X. Bernoulli Law Problems

A. Capillary Rise





Because of surface tension, fluid wetting pair of glass plates rises to a height ξ(r) determined by the surface tension γ and local distance between plates. Experiment from film "Surface Tension in Fluid Mechanics" (Reference 9, Appendix C).

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$$\frac{\partial}{\partial t}\theta = 0, \bar{v} = 0, \bar{g} = -g\,\bar{i}_z, \mathcal{E} = 0$$

$$P_c = P_b + \rho g \xi$$

$$P_a = P_d = P_c$$

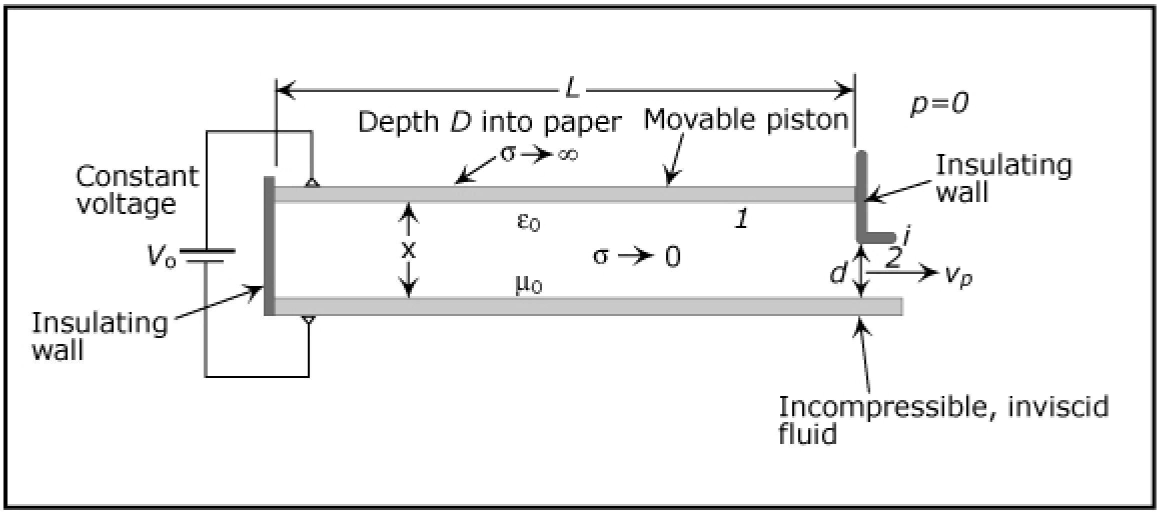
$$P_b - P_a + \gamma\left(\frac{1}{R} + \frac{1}{\infty}\right) = 0$$

$$R = \frac{\alpha r}{2}$$

$$P_a = P_b + \rho g \xi \Rightarrow P_a - P_b = \rho g \xi = \frac{2\gamma}{\alpha r}$$

$$\xi = \frac{2\gamma}{\rho g \alpha r}$$

B. Electrically Driven Rocket



$$pLD + f_x^e = 0$$

$$C = \frac{\varepsilon_0 LD}{x}, \;\; f_x^e = \frac{1}{2} V_0^2 \frac{dC}{dx} = -\frac{1}{2}\frac{V_0^2 \varepsilon_0 LD}{x^2}$$

$$p = \frac{-f_x^e}{LD} = \frac{1}{2}\frac{V_0^2 \varepsilon_0 \cancel{LD}}{x^2 \cancel{LD}} = \frac{1}{2}\varepsilon_0 \frac{V_0^2}{x^2}$$

Another Way:

$$p - T_{xx}^e = 0; \; T_{xx}^e = \frac{1}{2}\varepsilon_0 \left(\frac{V_0}{x}\right)^2$$

$$p = T_{xx}^e = \frac{1}{2}\varepsilon_0 \frac{V_0^2}{x^2}$$

$$p_1 + \frac{1}{2}\rho v_1^2 = \cancelto{0}{p_2} + \frac{1}{2}\rho v_2^2$$

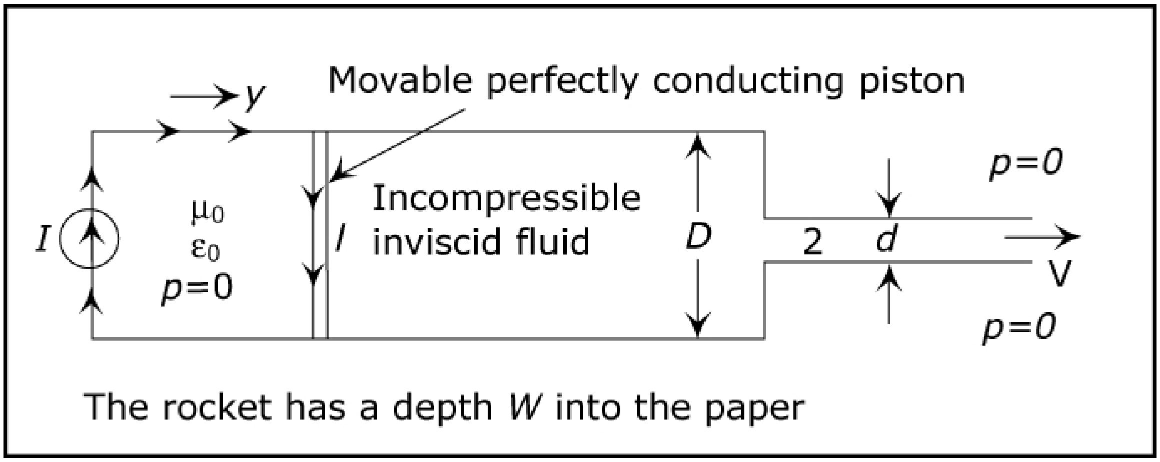
$$v_1 L\cancel{D} = v_2 d\cancel{D}$$

$$p_1 = \frac{1}{2}\varepsilon_0 \frac{V_0^2}{x^2} = \frac{1}{2}\rho v_2^2 \left(1 - \left(\frac{d}{L}\right)^2\right)$$

$$v_2 = V_p = \left\{ \frac{\varepsilon_0 V_0^2}{\rho x^2 \left[1 - \left(\frac{d}{L} \right)^2 \right]} \right\}^{\frac{1}{2}}$$

$$\text{Thrust} = V_p \frac{dM}{dt} = V_p^2 \rho \, dD$$

C. Magnetically Driven Rocket



The rocket has a depth *W* into the paper

$$+p + T_{yy}^{e} = 0$$

$$T_{yy} = -\frac{\mu_0}{2} H_z^2 = -\frac{\mu_0}{2} \left(\frac{I}{w} \right)^2$$

$$p = \frac{\mu_0}{2} \left(\frac{I}{w} \right)^2$$

$$p_1 + \frac{1}{2} \rho v_1^2 = \cancelto{0}{p_2} + \frac{1}{2} \rho v_2^2$$

$$p_1 = \frac{\mu_0}{2} \left(\frac{I}{w} \right)^2, \;\; v_1 D w = v_2 d w$$

$$\frac{\mu_0}{2} \left(\frac{I}{w} \right)^2 = \frac{1}{2} \rho v_2^2 \left(1 - \left(\frac{d}{D} \right)^2 \right)$$

$$v_2 = V = \left\{ \frac{\frac{\mu_0}{\rho} \left(\frac{I}{w} \right)^2}{\left[1 - \left(\frac{d}{D} \right)^2 \right]} \right\}^{\frac{1}{2}}$$

$$\text{Thrust} = V\frac{dM}{dt} = V\rho V dw = \rho V^2 dw$$

D. Dielectric Liquid Rise

1. Kelvin Polarization Force Density

$$\bar{F} = \left(\bar{P} \cdot \nabla\right)\bar{E}, \quad T_{ij} = E_i D_j - \frac{1}{2}\delta_{ij}\varepsilon_0 E_k E_k$$

$$\bar{P} = \left(\varepsilon - \varepsilon_0\right)\bar{E}$$

$$\left(\bar{P} \cdot \nabla\right)\bar{E} = \left(\varepsilon - \varepsilon_0\right)\left(\bar{E} \cdot \nabla\right)\bar{E}$$

$$\left(\nabla \times \cancelto{0}{\bar{E}}\right) \times \bar{E} = \left(\bar{E} \cdot \nabla\right)\bar{E} - \frac{1}{2}\nabla\left(\bar{E} \cdot \bar{E}\right)$$

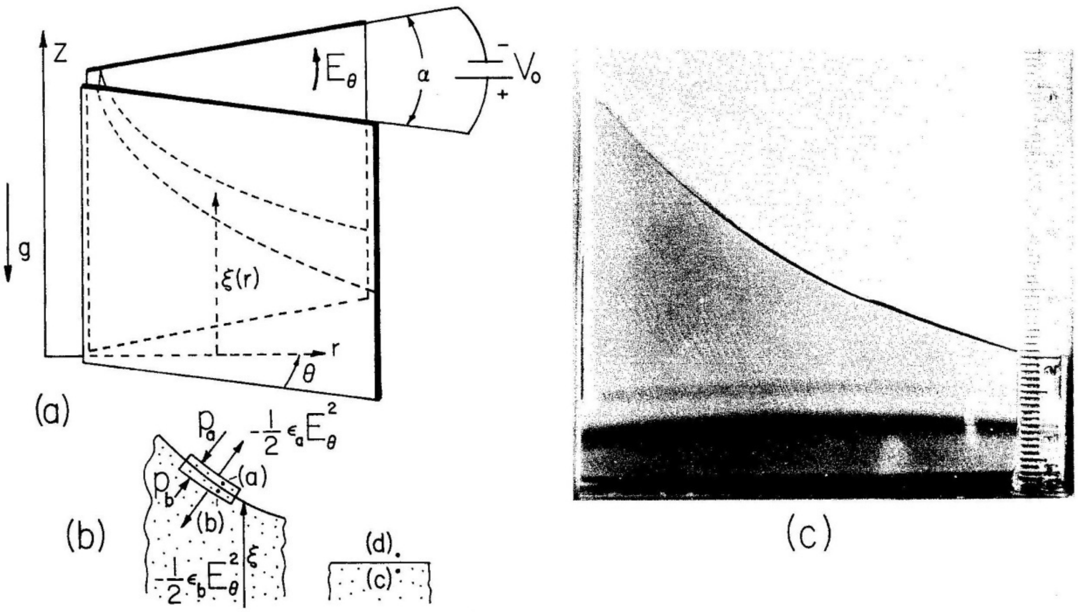
$$\left(\bar{E} \cdot \nabla\right)\bar{E} = \frac{1}{2}\nabla\left(\bar{E} \cdot \bar{E}\right)$$

$$\left(\bar{P} \cdot \nabla\right)\bar{E} = \frac{\left(\varepsilon - \varepsilon_0\right)}{2}\nabla\left(\bar{E} \cdot \bar{E}\right) = \nabla\left[\frac{1}{2}\left(\varepsilon - \varepsilon_0\right)\bar{E} \cdot \bar{E}\right] \quad \text{if } \varepsilon \text{ constant}$$

$$\bar{F} = -\nabla\mathcal{E} = \nabla\left[\frac{1}{2}\left(\varepsilon - \varepsilon_0\right)\bar{E} \cdot \bar{E}\right]$$

$$\mathcal{E} = -\frac{1}{2}\left(\varepsilon - \varepsilon_0\right)\bar{E} \cdot \bar{E}$$

$$P + \rho g z - \frac{1}{2}\left(\varepsilon - \varepsilon_0\right)\bar{E} \cdot \bar{E} = \text{constant}$$



(a) Diverging conducting plates with potential difference V_o are immersed in dielectric liquid. (b) Interfacial stress balance. (c) From Reference 12, Appendix C; corn oil ($\varepsilon = 3.7\ \varepsilon_o$) rises in proportion to local E^2. Upper fluid is compressed nitrogen gas ($\varepsilon \simeq \varepsilon_o$) so that E can approach 10^7 V/m required to raise liquid several cm. To avoid free charge effects, fields are 400 Hz a-c. The fluid responds to the time-average stress.

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$$\bar{E} = \frac{V_0}{\alpha r}\bar{i}_\theta$$

$$P_a = P_d$$

$$P_b + \rho g \xi - \frac{1}{2}\frac{(\varepsilon - \varepsilon_0)V_0^2}{\alpha^2 r^2} = P_c$$

$$P_b - P_a + \left\| T_{nn} \right\| = 0 \qquad \gamma = 0 \text{ (negligible surface tension)}$$

$$P_c - P_d = 0$$

$$\left\| T_{nn} \right\| = \left\| -\frac{1}{2}\varepsilon_0 E_\theta^2 \right\| = 0 \qquad (E_\theta \text{ continuous across interface})$$

$$P_a = P_b = P_c = P_d$$

$$\xi = \frac{1}{2}\frac{(\varepsilon - \varepsilon_0)V_0^2}{\alpha^2 r^2 \rho g}$$

2. Korteweg-Helmholtz Force Density

$$\bar{F} = -\frac{1}{2}\bar{E}\bullet\bar{E}\,\nabla\varepsilon, \quad T_{ij} = \varepsilon E_i E_j - \frac{1}{2}\delta_{ij}\varepsilon E_K E_K$$

In fluid ε = constant, $F = -\nabla\mathcal{E} = 0 \Rightarrow \mathcal{E} = 0$

$$P + \rho g z = \text{constant}$$

$$\|T_{nn}\| = \left\| -\frac{1}{2}\varepsilon_0 E_\theta^2 \right\| = -\frac{1}{2}(\varepsilon_0 - \varepsilon)\left(\frac{V_0}{\alpha r}\right)^2$$

$$P_a = P_d$$

$$P_b - P_a + \|T_{nn}\| = 0$$

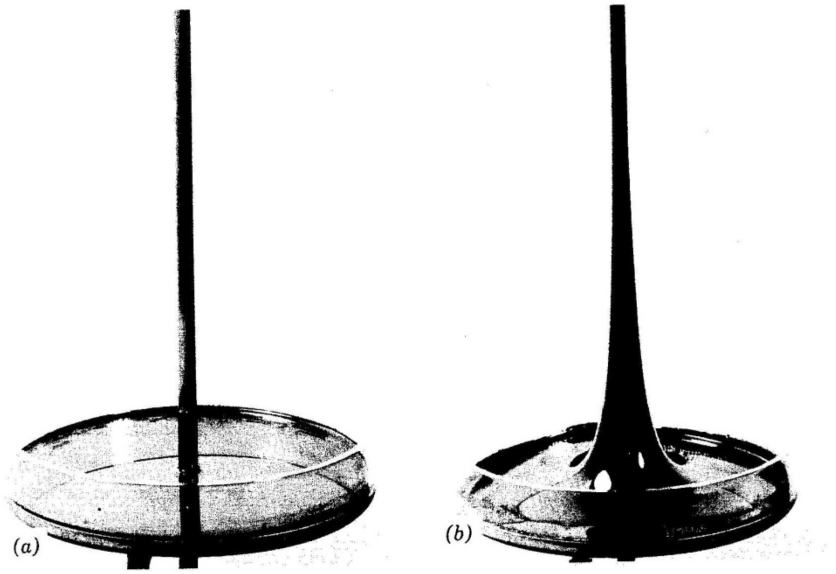
$$P_c - P_d = 0$$

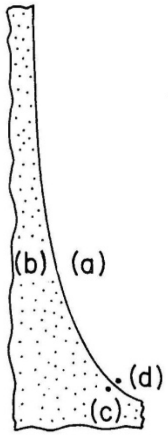
$$P_b + \rho g \xi = P_c$$

$$P_a - P_b = \rho g \xi = \|T_{nn}\| = -\frac{1}{2}(\varepsilon_0 - \varepsilon)\left(\frac{V_0}{\alpha r}\right)^2 = +\frac{1}{2}(\varepsilon - \varepsilon_0)\left(\frac{V_0}{\alpha r}\right)^2$$

$$\xi = \frac{1}{2}\frac{(\varepsilon - \varepsilon_0)V_0^2}{\alpha^2 r^2 \rho g}$$

E. Magnetic Fluid Rise in Tangential Magnetic Field





A magnetizable liquid is drawn upward around a current-carrying wire.
(Courtesy of AVCO Corporation, Space Systems Division.)

Courtesy of MIT Press. Used with permission.

$$\overline{H} = \frac{I}{2\pi r}\,\overline{i}_\theta$$

$$\overline{F} = \mu_0\left(\overline{M}\cdot\nabla\right)\overline{H}, \quad T_{ij} = H_i B_j - \frac{1}{2}\mu_0 \delta_{ij} H_K H_K$$

1. Linearly Magnetizable

$$\overline{B} = \mu_0\left(\overline{H} + \overline{M}\right) = \mu\overline{H} \;\Rightarrow\; \mu_0 M = \left(\mu - \mu_0\right)\overline{H}$$

$$\overline{F} = \left(\mu - \mu_0\right)\left(\overline{H}\cdot\nabla\right)\overline{H} = \left(\mu - \mu_0\right)\left[\cancelto{0}{\left(\nabla\times\overline{H}\right)}\times\overline{H} + \frac{1}{2}\nabla\left(\overline{H}\cdot\overline{H}\right)\right]$$

$$\overline{F} = \frac{\left(\mu - \mu_0\right)}{2}\nabla\left(\overline{H}\cdot\overline{H}\right) = \nabla\left[\frac{\left(\mu - \mu_0\right)}{2}\overline{H}\cdot\overline{H}\right] \quad \text{if } \mu = \text{constant}$$

$$= -\nabla\mathcal{E}$$

$$\mathcal{E} = -\frac{1}{2}\left(\mu - \mu_0\right)\overline{H}\cdot\overline{H}$$

$$P + \rho g z - \frac{1}{2}\left(\mu - \mu_0\right)\overline{H}\cdot\overline{H} = \text{constant}$$

$$P_a = P_d$$

$$P_c = P_d$$

$$P_b - P_a + \left\| T_{nn} \right\| = 0$$

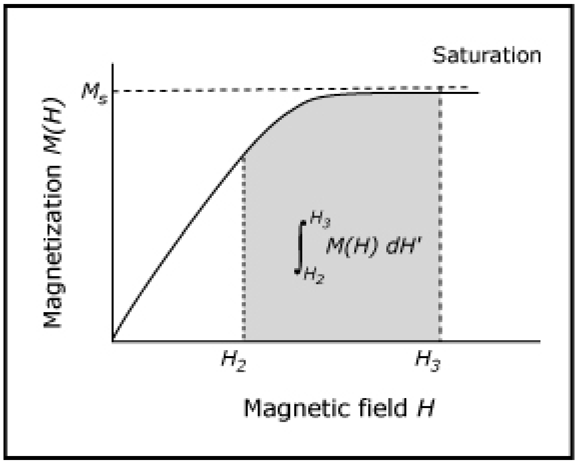
$$\left\| T_{nn} \right\| = \left\| -\frac{1}{2}\mu_0\, H_\theta^2 \right\| = 0$$

$$P_a = P_b = P_c = P_d$$

$$P_b + \rho g \xi - \frac{1}{2}\left(\mu - \mu_0\right)\left(\frac{I}{2\pi r}\right)^2 = P_c$$

$$\xi = \frac{1}{2}\frac{\left(\mu - \mu_0\right)}{\rho g}\left(\frac{I}{2\pi r}\right)^2$$

2. Nonlinear Magnetization Characteristics



$$\overline{M} = M(H)\,\overline{H}/H \qquad \left(\overline{M} \parallel \overline{H}\right)$$

$$\overline{F} = \mu_0\left(\overline{M}\cdot\nabla\right)\overline{H} = \mu_0\frac{M}{H}\left(\overline{H}\cdot\nabla\right)\overline{H}$$

$$\left(\overline{H}\cdot\nabla\right)\overline{H} = \overset{0}{\cancel{\left(\nabla\times\overline{H}\right)}}\times\overline{H} + \frac{1}{2}\nabla\left(\overline{H}\cdot\overline{H}\right)$$

$$\overline{F} = \frac{\mu_0 M}{H}\frac{1}{2}\nabla H^2 = \mu_0 M\nabla H = \nabla\int_0^H \mu_0 M(H)\,dH = -\nabla\mathcal{E}$$

$$\mathcal{E} = -\int_0^H \mu_0 M(H)\,dH = -\mu_0\widetilde{M}\,H; \quad \widetilde{M} = \frac{1}{H}\int_0^H M(H)\,dH$$

Special case:

Linear Material:

$$\mu_0 M = \left(\mu - \mu_0\right)H$$

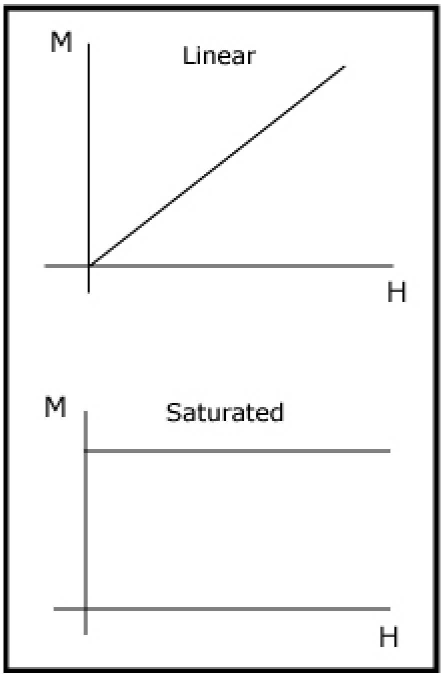
$$\mathcal{E} = -\frac{1}{2}\left(\mu - \mu_0\right)H^2 = -\frac{1}{2}\mu_0 MH$$

Saturated Material:

$$\mu_0 M = \text{constant}$$

$$\mathcal{E} = -\mu_0 MH$$

$$P + \rho gz - \mathcal{E} = \text{constant}$$



$$P_a = P_d$$
$$P_c = P_d$$

$$P_b - P_a + \left\| T_{nn} \right\| = 0$$

$$P_b + \rho g \xi + \mathcal{E}_b = P_c + \cancelto{0}{\mathcal{E}_c}$$

$$T_{ij} = H_i B_j - \frac{1}{2}\delta_{ij}\mu_0 H_K H_K$$

$$T_{nn} = -\frac{1}{2}\mu_0 H_\theta^2 \Rightarrow \left\| T_{nn} \right\| = 0$$

$$P_a = P_b = P_c = P_d$$

$$\xi = \frac{-\mathcal{E}_b}{\rho g} = \frac{\int_0^H \mu_0 M(H)\, dH}{\rho g} = \frac{\mu_0 \widetilde{M}\, H}{\rho g}$$

Linear Material:

$$\mathcal{E}_b = -\frac{1}{2}\left(\mu - \mu_0\right)\left(\frac{I}{2\pi r}\right)^2$$

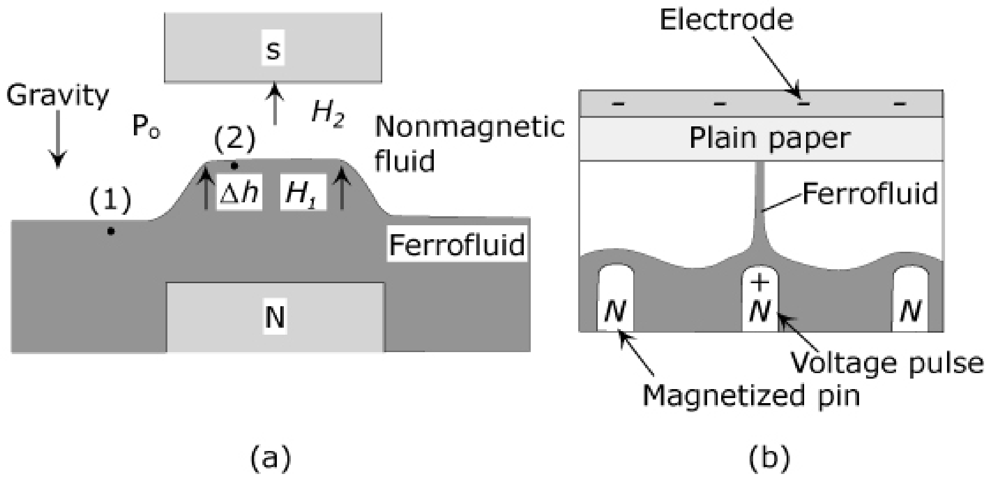
$$\xi = \frac{1}{2}\frac{\left(\mu - \mu_0\right) I^2}{\left(2\pi r\right)^2 \rho g}$$

Saturated Material:

$$\mathcal{E}_b = -\mu_0 M H$$

$$\xi = \frac{\mu_0 M H}{\rho g} = \frac{\mu_0 M I}{2\pi\rho g r}$$

F. Magnetic Fluid Rise in Normal Field



(a) Uniform magnetic field imposed normal to a free ferrofluid interface. The fluid magnetization is assumed smaller than required to produce the normal field instability.
(b) Printing process utilizing a wavy magnetic-fluid surface established in a spatially periodic magnetic field. Printing occurs on command, in which an electric pulse is made to eject a droplet of the fluid, which serves as the ink, from a surface peak. The sample of printing shown results from the simultaneous ejection of droplets from selected peaks along multiple rows of droplets onto a moving page.

$$P_1 = P_2 + \rho g \Delta h + \mathcal{E} = P_2 + \rho g \Delta h - \mu_0 \widetilde{M} H_1$$

$$P_1 = P_0$$

$$P_2 - P_0 + \|T_{nn}\| = 0$$

$$T_{nn} = H_x B_x - \frac{1}{2}\mu_0 H_x^2$$

$$\|T_{nn}\| = \frac{\mu_0}{2} H_2^2 - \mu_0 H_1 \left(H_1 + M\right) + \frac{1}{2}\mu_0 H_1^2$$

$$\cancel{\mu_0} H_2 = \cancel{\mu_0}\left(H_1 + M\right)$$

$$\|T_{nn}\| = \frac{\mu_0}{2}\left(H_1 + M\right)^2 - \mu_0 H_1 \left(H_1 + M\right) + \frac{1}{2}\mu_0 H_1^2$$

$$= \frac{\mu_0 M^2}{2}$$

$$P_2 = P_0 - \frac{\mu_0 M^2}{2}$$

$$P_1 = P_0$$

$$P_1 - P_2 = \frac{\mu_0 M^2}{2} = \rho g \Delta h - \mu_0 \widetilde{M} H_1$$

$$\Delta h = \frac{\mu_0}{\rho g}\left[\frac{M^2}{2} + \widetilde{M} H_1\right]$$

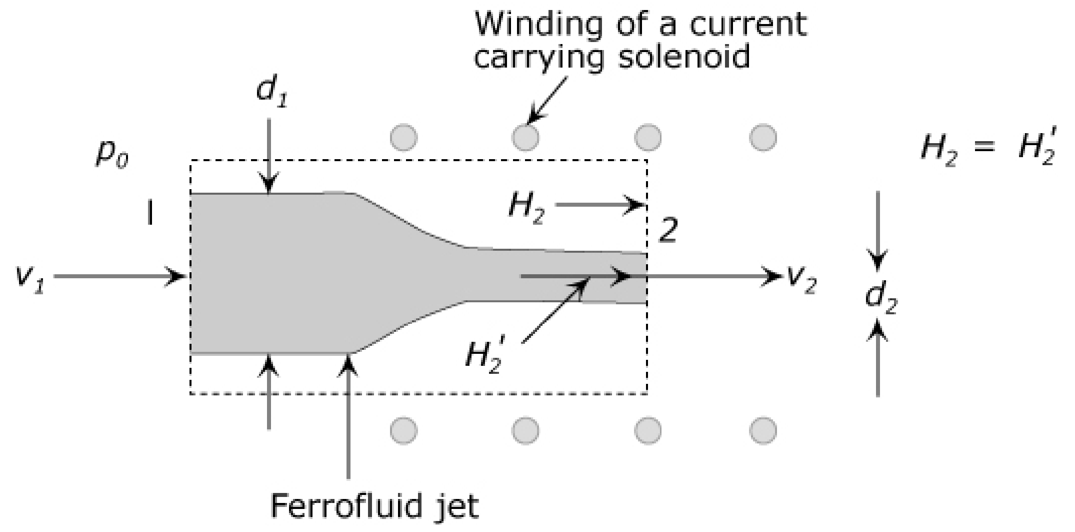
Linear Material:

$$\widetilde{M} = \frac{M}{2} \Rightarrow \Delta h = \frac{\mu_0}{\rho g}\frac{M}{2}\left[M + H_1\right] = \frac{M B_1}{2 \rho g}$$

Saturated Material:

$$\widetilde{M} = M \Rightarrow \Delta h = \frac{\mu_0}{\rho g}\frac{M}{2}\left[M + 2H_1\right]$$

G. Magnetic Nozzle



A free jet of magnetic fluid changes cross section and velocity in flowing through this magnetic nozzle.

$$P_1 + \frac{1}{2}\rho v_1^2 + 0 = P_2 + \frac{1}{2}\rho v_2^2 - \mu_0 \tilde{M}\, H_2$$

$$P_1 = P_0$$

$$P_2 = P_0$$

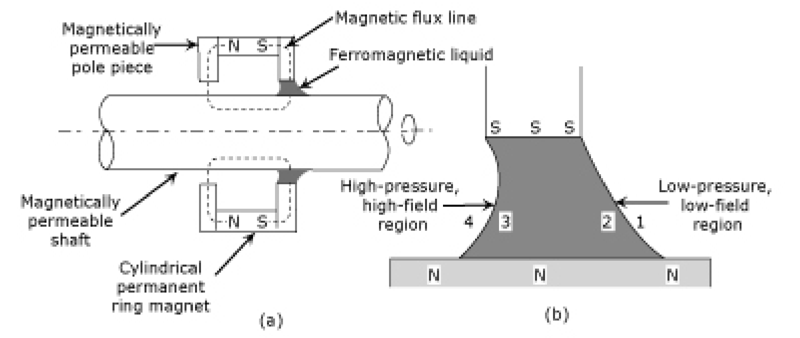
$$v_2^2 - v_1^2 = \frac{2\mu_0}{\rho}\tilde{M}\, H_2$$

$$\frac{\pi d_1^2}{4} v_1 = \frac{\pi d_2^2}{4} v_2 \Rightarrow \frac{d_1}{d_2} = \sqrt{\frac{v_2}{v_1}}$$

$$v_2 = \left[v_1^2 + \frac{2\mu_0}{\rho}\tilde{M}\, H_2 \right]^{1/2} = v_1 \left[1 + \frac{2\mu_0}{\rho v_1^2}\tilde{M}\, H_2 \right]^{1/2}$$

$$\frac{d_1}{d_2} = \left[1 + \frac{2\mu_0}{\rho v_1^2}\tilde{M}\, H_2 \right]^{1/4}$$

H. Magnetic Fluid Rotary Shaft Seal



(a) Components of a magnetic fluid rotary shaft seal.
(b) Sketch to analyze the pressure supported across one stage of the seal.

$$P_3 - \mu_0 \tilde{M}_3 H_3 = P_2 - \mu_0 \tilde{M}_2 H_2$$

$$P_4 - P_3 + \|T_{nn}\|_{3,4} = 0$$

$$P_2 - P_1 + \|T_{nn}\|_{1,2} = 0$$

Assume H tangential to interface

$$T_{nn} = -\frac{1}{2}\mu_0 H_t^2 \Rightarrow \|T_{nn}\|_{\substack{1,2\\3,4}} = 0$$

$$P_1 = P_2$$

$$P_4 = P_3$$

$$P_3 - P_2 = P_4 - P_1 = \Delta P = \mu_0\left(\widetilde{M}_3 H_3 - \widetilde{M}_2 H_2\right)$$

$$= \mu_0 \int_{H_2}^{H_3} M dH$$

In a well designed seal $H_2 \ll H_3$

$$\Delta P \approx \mu_0 \widetilde{M}\, H$$

Typical numbers:

$$\mu_0 M = 700\ G = .07T$$

$$\mu_0 H = 18,000\ G = 1.8T$$

$$\Delta P = \frac{(\mu_0 M)(\mu_0 H)}{\mu_0} = \frac{.07(1.8)}{4\pi \times 10^{-7}} = 10^5\ N/m^2$$

$$= 100\ \text{kPascals}$$

$$= 1\ \text{Atmosphere}$$

XI. Force on a Body in a Magnetic Fluid

$$p + \rho g z - \mu_0 \widetilde{M}\, H = \text{constant} = C$$

$$p = \mu_0 \widetilde{M}\, H - \rho g z + C$$

$$\bar{f}_p = -\oint_S p\, \bar{n}\, da = -\oint_S \mu_0 \widetilde{M}\, H\, \bar{n}\, da + \underbrace{\oint_S \rho g z\, \bar{n}}_{\substack{\text{buoyancy} \\ \text{effect}}} da + \cancelto{0}{\oint_S} C\, \bar{n}\, da$$

$$\oint_S C\, \bar{n}\, da = \int_V \underbrace{\nabla C}_{0} dV$$

Magnetically Saturated: $\widetilde{M} = M = \text{constant}$

$$\bar{f}_M = -\oint_S \mu_0 \widetilde{M}\, H\, \bar{n}\, da = -\mu_0 M \oint_S H\, \bar{n}\, da = -\mu_0 M \int \nabla H dV$$

Magnetically Linear: $\widetilde{M} = \frac{1}{2} M = \frac{1}{2}\frac{(\mu - \mu_0)}{\mu_0} H$

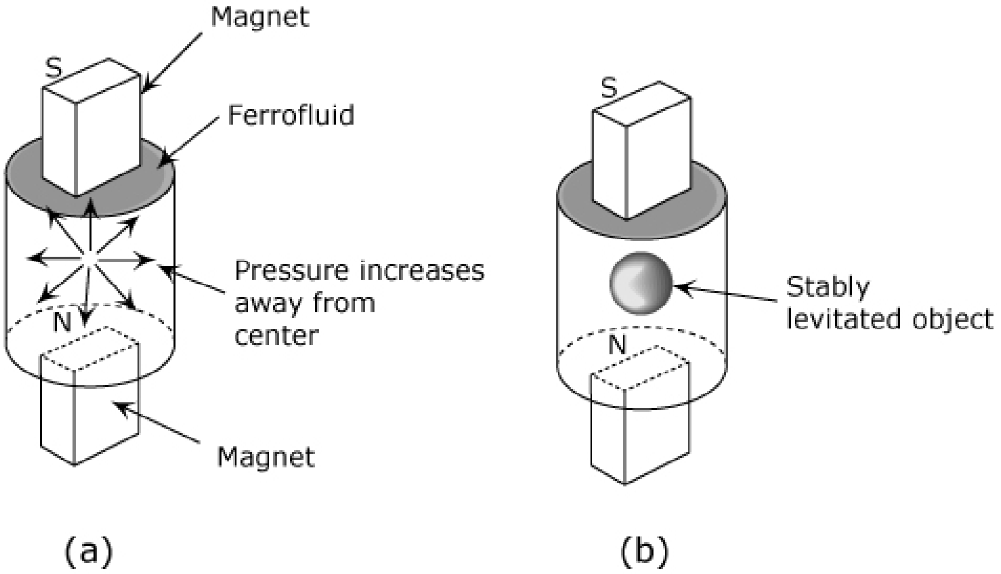
$$\bar{f}_M = -\oint_S \mu_0 \widetilde{M}\, H\, \bar{n}\, da = -\int_S \frac{1}{\cancel{2}}\mu_0 \frac{(\mu - \mu_0)}{\cancel{\mu_0}} H^2 \bar{n}\, da = -\int_V \nabla\left(\frac{1}{2}(\mu - \mu_0) H^2\right) dV$$

A. Non-magnetic Body

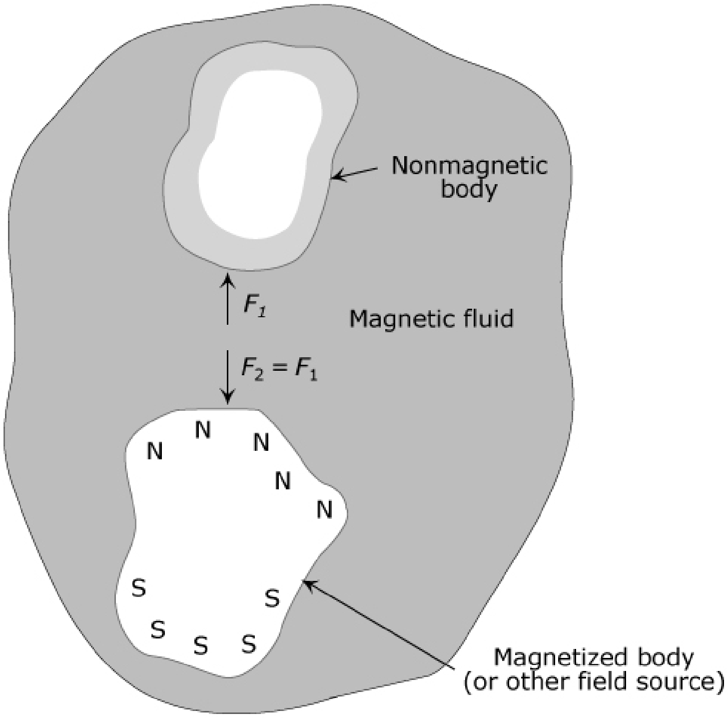
$$f_{body} = \text{buoyant weight} + f_M$$

f_M opposite to direction of increasing H

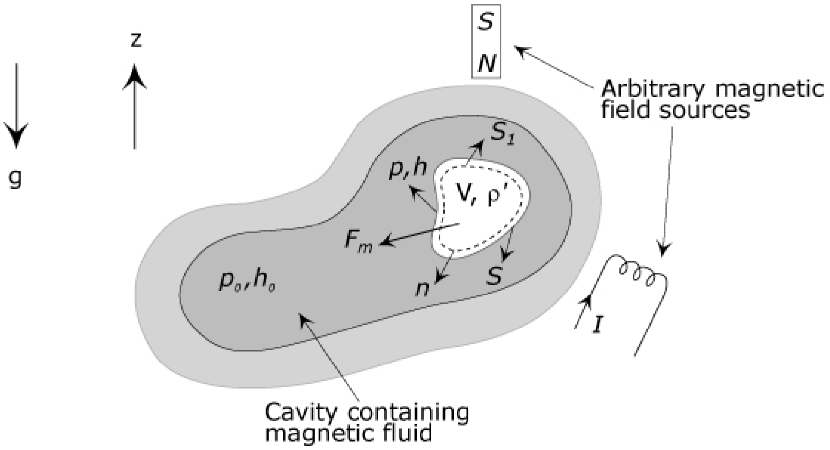
Non-magnetic body moves towards weak field region
(Sink-Float Separation)



Levitation of a nonmagnetic body in a magnetized fluid is another demonstration of the expanded Bernoulli relation. (a) Pressure in the fluid is lowest at the center and increases with distance from it. (b) When a nonmagnetic object such as a glass marble is placed in the container, it moves to the center and remains there in equilibrium.



Mutual repulsion of a magnet and a nonmagnetic object when both are immersed in a ferrofluid.



Development of the generalized expression for the force acting on an arbitrary magnetized or unmagnetized body immersed in a magnetic fluid and subjected to an arbitrary magnetic field.