6.642 Continuum Electromechanics
Fall 2008

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Problem Set 3 - Solutions

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Problem 1

a)

$\hat{\sigma}_0 = \sigma_0; \quad \hat{\Phi}_0 = +j\Phi_0.$

b)

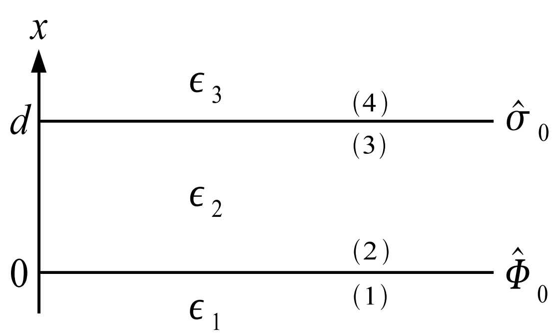


Figure 1: A three dielectric layer system with potential constrained at $x = 0$ and surface charge density constrained at $x = d$ (Image by MIT OpenCourseWare.)

$$x > d \quad \begin{bmatrix} \hat{E}_x(x=\infty) \\ \hat{E}_{x4} \end{bmatrix} = -k \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{\Phi}(x=\infty) \\ \hat{\Phi}_4 \end{bmatrix}$$

$$d > x > 0 \quad \begin{bmatrix} \hat{E}_{x3} \\ \hat{E}_{x2} \end{bmatrix} = -k \begin{bmatrix} \coth kd & -\frac{1}{\sinh kd} \\ \frac{1}{\sinh kd} & -\coth kd \end{bmatrix} \begin{bmatrix} \hat{\Phi}_3 \\ \hat{\Phi}_2 \end{bmatrix}$$

$$0 > x \quad \begin{bmatrix} \hat{E}_{x1} \\ \hat{E}_x(x=-\infty) \end{bmatrix} = -k \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{\Phi}(x=0) \\ \hat{\Phi}(x=-\infty) \end{bmatrix}$$

$$\hat{\Phi}_1 = \hat{\Phi}_2 = \hat{\Phi}_0$$

$$\hat{\Phi}_3 = \hat{\Phi}_4$$

$$\hat{\Phi}(x=\infty) = \hat{\Phi}(x=-\infty) = 0$$

$$\hat{E}_{x1} = -k\hat{\Phi}_0$$

$$\hat{E}_{x2} = -k\left(\frac{\hat{\Phi}_3}{\sinh kd} - \coth kd\hat{\Phi}_0\right)$$

$$\hat{E}_{x3} = -k\left(\coth kd\hat{\Phi}_3 - \frac{\hat{\Phi}_0}{\sinh kd}\right)$$

$$\hat{E}_{x4} = k\hat{\Phi}_4$$

$$\hat{E}_{z3} = \hat{E}_{z4} = jk\hat{\Phi}_4$$

BC. at $x = d \quad \epsilon_3\hat{E}_{x4} - \epsilon_2\hat{E}_{x3} = \hat{\sigma}_0; \quad \mathbf{E} = -\nabla\Phi$

$$\epsilon_3(+k)\hat{\Phi}_4 + \epsilon_2 k\left(\coth kd\hat{\Phi}_3 - \frac{1}{\sinh kd}\hat{\Phi}_0\right) = \hat{\sigma}_0$$

$$\hat{\Phi}_3 = \hat{\Phi}_4 = \frac{\hat{\sigma}_0 + \frac{\epsilon_2 k\hat{\Phi}_0}{\sinh kd}}{\epsilon_2 k\coth kd + \epsilon_3 k}$$

c)

at $x = 0 \quad \epsilon_2\hat{E}_{x2} - \epsilon_1\hat{E}_{x1} = \hat{\sigma}'$

$$\hat{\sigma}' = -k\epsilon_2\left(\frac{\hat{\Phi}_3}{\sinh kd} - \coth kd\hat{\Phi}_0\right) + k\epsilon_1\hat{\Phi}_0$$

$$= \left[(k\epsilon_1 + k\epsilon_2\coth kd) - \left(\frac{k\epsilon_2}{\sinh kd}\right)^2 \cdot \frac{1}{k\epsilon_3 + k\epsilon_2\coth kd}\right]\hat{\Phi}_0 - \frac{\epsilon_2 k\hat{\sigma}_0}{\sinh kd(k\epsilon_3 + k\epsilon_2\coth kd)}$$

d)

at $x = d \quad \mathbf{E} = -\nabla\Phi \Rightarrow E_z = -\frac{\partial\Phi}{\partial z} \Rightarrow \hat{E}_z = jk\hat{\Phi}$

$$\hat{E}_{z3} = \hat{E}_{z4} = jk\hat{\Phi}_3$$

$$\frac{f_x}{\text{Area}} = T_{xx}\cdot n_x|_{(4)} + T_{xx}\cdot n_x|_{(3)} = T_{xx(4)} - T_{xx(3)}$$

$$= -\frac{\epsilon_2}{2}(E_{x3}^2 - E_{z3}^2) + \frac{\epsilon_3}{2}(E_{x4}^2 - E_{z4}^2)$$

$$\frac{< f_x > |_z}{\text{Area}} = -\frac{\epsilon_2}{4}(|\hat{E}_{x3}|^2 - |\hat{E}_{z3}|^2) + \frac{\epsilon_3}{4}(|\hat{E}_{x4}|^2 - |\hat{E}_{z4}|^2)$$

$$= -\frac{\epsilon_2}{4(\epsilon_2\coth kd + \epsilon_3)^2}\left[(\coth^2 kd - 1)\sigma_0^2 + \frac{k^2\Phi_0^2}{\sinh^2 kd}(\epsilon_3^2 - \epsilon_2^2)\right]$$

$$\frac{f_z}{\text{Area}} = T_{zx}n_x|_{(4)} + T_{zx}n_x|_{(3)}$$

$$= -\epsilon_2 E_{x3}E_{z3} + \epsilon_3 E_{x4}E_{z4} = E_{z3}\cdot\sigma_0$$

$$\frac{< f_z > |_z}{\text{Area}} = \frac{1}{2}\Re\{\epsilon_3\hat{E}_{x4}\hat{E}_{z4}^* - \epsilon_2\hat{E}_{x3}\hat{E}_{z3}^*\}$$

$$= \frac{1}{2}\Re\{jk\hat{\Phi}_4\cdot\hat{\sigma}_0^*\} = -\frac{\epsilon_2 k\sigma_0\Phi_0}{2(\epsilon_2\coth kd + \epsilon_3)\sinh kd}$$

Problem 2

a)

$$\nabla^2\mathbf{H} = \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rH_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 H_r}{\partial\phi^2} - \frac{2}{r^2}\frac{\partial H_\phi}{\partial\phi} + \frac{\partial^2 H_z}{\partial z^2}\right]\mathbf{i_r}$$

$$+\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rH_\phi)0}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 H_r}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial H_r}{\partial\phi} + \frac{\partial^2 H_\phi}{\partial z^2}\right]\mathbf{i}_\phi$$

$$+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial\phi^2} + \frac{\partial^2 H_z}{\partial z^2}\right]\mathbf{i_z}$$

$$\mathbf{H}(r,\phi,z) = H_r(r,\phi,z)\mathbf{i_r} + H_\phi(r,\phi,z)\mathbf{i}_\phi + H_z(r,\phi,z)\mathbf{i_z}$$

For $\mathbf{H}(r) = H_z(r)\mathbf{i_z}$

$$\nabla^2\mathbf{H} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right), \quad \mathbf{H} = \Re[\hat{H}_z(r)e^{j\omega t}]\mathbf{i_z}$$

$$\nabla^2\mathbf{H} = \mu\sigma\frac{\partial\mathbf{H}}{\partial t} \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{d\hat{H}_z}{dr}\right) = j\omega\mu\sigma\hat{H}_z$$

b)

Bessel's Equation: $r\frac{d}{dr}\left(r\frac{d\hat{H}_z}{dr}\right) = -(k^2r^2 - n^2)\hat{H}_z = 0$

Multiply (a) by r^2:

$$r\frac{d}{dr}\left(r\frac{d\hat{H}_z}{dr}\right) = j\omega\mu\sigma r^2\hat{H}_z \Rightarrow k^2 = -j\omega\mu\sigma, \quad n = 0$$

$$k = \pm j\sqrt{j}\sqrt{\omega\mu\sigma} = \pm\frac{j(1+j)}{\sqrt{2}}\sqrt{\omega\mu\sigma} = \pm\frac{j-1}{\delta}, \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$\hat{H}_z = c_1 J_0(kr) + c_2 Y_0(kr)$$

B.C.: $\hat{H}_z(r=0)$ is finite $\Rightarrow c_2 = 0$

$$\hat{H}_z(r=R) = \hat{H}_0$$

$$\hat{H}_z = \hat{H}_0\frac{J_0(kr)}{J_0(kR)}$$

Since $J_0(kr)$ has a series form of even powers of (kr), either sign can be used for k.

c)

$$\nabla\times\mathbf{H} = -\mathbf{i}_\phi\frac{\partial H_z}{\partial r} = J_\phi\mathbf{i}_\phi \Rightarrow J_\phi = -\frac{\partial H_z}{\partial r} \Rightarrow \hat{J}_\phi = -\frac{d\hat{H}_z}{dr}$$

$$\frac{dJ_0(z)}{dz} = -J_1(z) \Rightarrow \hat{J}_\phi = -\frac{\hat{H}_0 k J_1(kr)}{J_0(kR)}$$

d)

Mathematica Program to Plot $H_z(r, t = 0)/H_0$ **and** $J_\phi(r, t = 0)/(H_0 l\delta)$

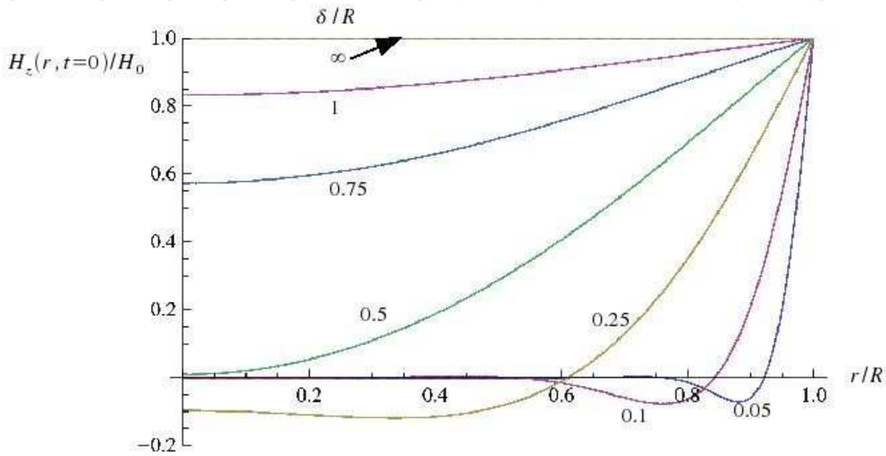
Clear[Hzn, Jphin]
Hzn[rn_, deltan_] = Re[BesselJ[0, (1 − I) ∗ rn/deltan]/BesselJ[0, (1 − I)/deltan]]

$\text{Re}\left[\frac{\text{BesselJ}\left[0, \frac{(1-i)\text{rn}}{\text{deltan}}\right]}{\text{BesselJ}\left[0, \frac{1-i}{\text{deltan}}\right]}\right]$

Jphin[rn_, deltan_] =
Re[−(1 − I) ∗ BesselJ[1, (1 − I) ∗ rn/deltan]/BesselJ[0, (1 − I)/deltan]]

$\text{Re}\left[-\frac{(1-i)\text{BesselJ}\left[1, \frac{(1-i)\text{rn}}{\text{deltan}}\right]}{\text{BesselJ}\left[0, \frac{1-i}{\text{deltan}}\right]}\right]$

Plot[{Hzn[rn, 0.05], Hzn[rn, 0.1], Hzn[rn, 0.25], Hzn[rn, 0.5], Hzn[rn, 0.75], Hzn[rn, 1], Hzn[rn, Infinity]}, {rn, 0, 1}, PlotRange → {−.2, 1}]



Plot[{Jphin[rn, 0.05], Jphin[rn, 0.1], Jphin[rn, 0.25], Jphin[rn, 0.5], Jphin[rn, 0.75], Jphin[rn, 1], Jphin[rn, Infinity]}, {rn, 0, 1}, PlotRange → {−.2, 1}]