MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

Problem Set No. 8 6.632 Electromagnetic Wave Theory
Spring Term 2003

Reading assignment: Section 4.1-4.3, J. A. Kong, "*Electromagnetic Wave Theory*"

Problem P8.1 In the Čerenkov radiation, the total energy radiated by a cylinder of path l and radius ρ is

$$S_\rho = \frac{\mu q^2 l}{4\pi} \int_0^\infty d\omega\, \omega \left(1 - \frac{1}{n^2\beta^2}\right)$$

So, the energy lost per unit length per unit frequency band is

$$\frac{dS_\rho}{dl\, d\omega} = \frac{\mu q^2}{4\pi}\omega \left(1 - \frac{1}{n^2\beta^2}\right)$$

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(a) By $E_{photon} = \hbar\omega$ and $\frac{d\omega}{d\lambda} = \frac{2\pi c}{\lambda^2}$, show that the number of photon radiated on unit path at wave length λ is

$$\frac{d^2 N}{dl\, d\lambda} = \frac{q^2 c}{2\lambda^2\hbar}\mu(1 - \frac{1}{n^2\beta^2})$$

and show the frequently used formula $\frac{dN}{dl} \propto \frac{d\lambda}{\lambda^2}\sin^2\theta$, which gives the dependence of N on λ and θ.

(b) Gas Čerenkov detector is widely used in particle experiment. The refractive index of the gas n is typically 1.002. What will be the angle for the Čerenkov radiation in case of $\beta = 1$?

(c) Most energy is radiated by the waves in the band 350 nm$\sim$550 nm. How many photons can you get on unit path? In order to get 100 photons for the detector, how long is the path (l)? This is the size the detector should be. Note that the parameters are as follows: $\hbar = 6.63 \times 10^{-34}/(2\pi)\mathrm{J}\cdot\mathrm{s/rad}$, $q = 1.6 \times 10^{-19}\mathrm{C}$, $\beta = 1$.

Problem P8.2

(a) Find the far field electric and magnetic vectors due to an infinitely long line current source with $I(z) = I_0 e^{ik_z z}$, placed along the z axis in free space. Show that the electric and magnetic fields in the far field are

$$\overline{E}(\overline{r}) \approx \frac{I_0 k_\rho}{4\omega\epsilon}\sqrt{\frac{2}{\pi k_\rho \rho}}[\hat{\rho}k_z - \hat{z}k_\rho]e^{i(k_\rho\rho + k_z z - \frac{\pi}{4})}$$

$$\overline{H}(\overline{r}) \approx \hat{\phi}\frac{I_0 k_\rho}{4}\sqrt{\frac{2}{\pi k_\rho \rho}}e^{i(k_\rho\rho + k_z z - \frac{\pi}{4})}$$

(b) Evaluate the real part of the complex Poynting's vector in the far field. What happens if $k_z > k$?

(c) Determine the equi-phase surfaces (phase fronts) in the far field, both for $k_z < k$ and $k_z > k$. Is the real part of the Poynting's vector normal to the equi-phase surfaces?

Problem P8.3

The electric field produced by a particle with charge q moving at a velocity $v\hat{z}$ is

$$\overline{E}(\overline{r}) = \frac{-q}{8\pi\omega\epsilon}\left[\hat{z}k^2 + i\frac{\omega}{v}\nabla\right] H_0^{(1)}(k_\rho\rho)e^{i\omega z/v}$$

In the radiation zone, $k_\rho\rho \gg 1$, and $H_0^{(1)}(k_\rho\rho) \approx \sqrt{\frac{2}{i\pi k_\rho\rho}}e^{ik_\rho\rho}$.

(a) Write the electric field in the radiation zone.
(b) Write the wave vector $\overline{k}$.
(c) What is the range of speed v in which k_ρ is real?
(d) Specify the polarization of the wave.

Problem P8.4

The inner product or dot product between two column vector $\overline{A}$ and $\overline{B}$, each represented as a $n \times 1$ matrix a and b can be expressed as

$$\overline{A}\cdot\overline{B} = a^T b$$

and the outer product between $\overline{A}$ and $\overline{B}$ can be expressed as

$$\overline{A}\,\overline{B} = ab^T$$

Let $\overline{A} = (\hat{x} + \hat{y} + \hat{z})$, so whereas

$$\overline{A}\cdot\overline{A} = 3$$

is a scaler, the outer product of the two vector:

$$\overline{A}\,\overline{A}$$

gives

$$\begin{bmatrix} \hat{x}\hat{x} & \hat{x}\hat{y} & \hat{x}\hat{z} \\ \hat{y}\hat{x} & \hat{y}\hat{y} & \hat{y}\hat{z} \\ \hat{z}\hat{x} & \hat{z}\hat{y} & \hat{z}\hat{z} \end{bmatrix}$$

Each column of the resulting dyadic can be view as a vector and operators that operate on the dyadic operate in a column-wise fashion:

$$\nabla\cdot\begin{bmatrix} \hat{x}\hat{x} & \hat{x}\hat{y} & \hat{x}\hat{z} \\ \hat{y}\hat{x} & \hat{y}\hat{y} & \hat{y}\hat{z} \\ \hat{z}\hat{x} & \hat{z}\hat{y} & \hat{z}\hat{z} \end{bmatrix} = (\nabla\cdot\overline{A})\hat{x} + (\nabla\cdot\overline{A})\hat{y} + (\nabla\cdot\overline{A})\hat{z}$$

(i) Show that $\nabla\cdot\overline{\overline{I}} = \nabla$

(ii) By taking the divergence of the equation

$$\nabla\times\nabla\times\overline{\overline{G}} - k^2\overline{\overline{G}} = \overline{\overline{I}}\delta(\overline{r}-\overline{r}') \tag{P1}$$

show that

$$-k^2\nabla\cdot\overline{\overline{G}} = \nabla\delta(\overline{r}-\overline{r}')$$

(iii) Expand (P1), show that

$$(\nabla^2+k^2)\overline{\overline{G}} = -\overline{\overline{I}}\delta(\overline{r}-\overline{r}') + \nabla\nabla\cdot\overline{\overline{G}}$$

(iv) Combine part (ii) and (iii), show that

$$(\nabla^2+k^2)\overline{\overline{G}} = -\left(\overline{\overline{I}}+\frac{1}{k^2}\nabla\nabla\right)\delta(\overline{r}-\overline{r}')$$

and hence by writting $\overline{\overline{G}} = \left(\overline{\overline{I}}+\frac{1}{k^2}\nabla\nabla\right)g$, all we need to solve is the scaler equation $(\nabla^2+k^2)g = -\delta(\overline{r}-\overline{r}')$.

Problem P8.5

What is the differential equation that governs the one-dimensional scalar Green's function in free space $g(x,x')$? Show that the one-dimensional Green's function is

$$g(x,x') = \frac{ie^{ik|x-x'|}}{2k}$$

Problem P8.6

The magnetic field $\overline{H}$ and electric field $\overline{E}$ of a Hertzian dipole at very large distances ($kr \gg 1$) are

$$\overline{H} = -\hat{\phi}\frac{\omega k q\ell}{4\pi r}\sin\theta\cos(kr-\omega t)$$
$$\overline{E} = -\hat{\theta}\frac{k^2 q\ell}{4\pi\epsilon_o r}\sin\theta\cos(kr-\omega t)$$

(a) Find the Poynting's power density vector $\overline{S}$ as a function of time. What is the time-averaged power density vector $\left\langle\overline{S}\right\rangle$?

(b) By integrating the Poynting vector over the surface of a sphere of radius r, find the time-averaged power P radiated by the Hertzian dipole.

(c) The amplitude of the current in the Hertzian dipole is $I_o = \omega q$. By using $P = \frac{1}{2}I_o^2 R_{rad}$, find the radiation resistance R_{rad} of the Hertzian dipole.