Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Lecture Number 5
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Jeffrey H. Shapiro

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Reading: For coherent states and minimum uncertainty states:

- C.C. Gerry and P.L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2005) Sects. 3.1, 3.5, 3.6.
- R. Loudon, *The Quantum Theory of Light* (Oxford University Press, Oxford, 1973) chapter 7.
- L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995) Sects. 11.1–11.6.

Introduction

Today we continue our development of the quantum harmonic oscillator, with a primary focus on measurement statistics and the transition to the classical limit of noiseless oscillation. In particular, we'll work with the time-dependent annihilation operator,

$$\hat{a}(t) = \hat{a}e^{-j\omega t}, \quad \text{for } t \geq 0, \tag{1}$$

its quadrature components[1]

$$\hat{a}_1(t) \equiv \text{Re}[\hat{a}(t)] = \text{Re}(\hat{a}e^{-j\omega t}) \quad \text{and} \quad \hat{a}_2(t) \equiv \text{Im}[\hat{a}(t)] = \text{Im}(\hat{a}e^{-j\omega t}), \tag{2}$$

and the number operator

$$\hat{N} = \hat{a}^\dagger(t)\hat{a}(t) = \hat{a}^\dagger\hat{a}. \tag{3}$$

[1] There are three equivalent representations for a real-valued classical sinusoid, $x(t)$, of frequency ω: (1) the phasor (complex-amplitude) representation, $x(t) = \text{Re}(\mathbf{x}e^{-j\omega t})$, where $\mathbf{x}$ is a complex number; (2) the quadrature-component representation, $x(t) = x_c\cos(\omega t) + x_s\sin(\omega t)$, where x_c and x_s are real numbers; and (3) the amplitude and phase representation, $x(t) = A\cos(\omega t - \phi)$, where A is a non-negative real number and ϕ is a real number. Taking $\mathbf{x} = x_c + jx_s = Ae^{j\phi}$ establishes the connections between these representations. We are using the first two in our quantum treatment of the harmonic oscillator. There are subtleties—which we may go into later—in trying to use the amplitude and phase representation for the quantum harmonic oscillator.

In terms of the number operator's orthonormal eigenkets, $\{|n\rangle\}$, and associated eigenvalues, $\{n\}$, we have

$$\hat{N} = \sum_{n=0}^{\infty} n|n\rangle\langle n| \quad \text{and} \quad \hat{I} = \sum_{n=0}^{\infty} |n\rangle\langle n|, \tag{4}$$

as well as

$$\hat{a} = \sum_{n=1}^{\infty} \sqrt{n}\,|n-1\rangle\langle n| \quad \text{and} \quad \hat{a}^\dagger = \sum_{n=0}^{\infty} \sqrt{n+1}\,|n+1\rangle\langle n|, \tag{5}$$

which will also be of use in what follows. Although we will not make much use of the Hamiltonian in today's lecture, we note that its eigenket-eigenvalue expansion is

$$\hat{H} = \hbar\omega(\hat{N} + 1/2) = \sum_{n=0}^{\infty} \hbar\omega(n + 1/2)|n\rangle\langle n|, \tag{6}$$

where the minimum energy, $\hbar\omega/2$, which is associated with the zero-quantum (zero-photon) state $|0\rangle$, is called the zero-point energy. What we will develop today is very much in keeping with a basic principle of quantum mechanics: the *state* of a quantum system *and* the *measurement* that is made on that system determine the statistics of the resulting measurement outcomes. We will see that the zero-point energy plays a key role in the quadrature-measurement statistics.

Quadrature-Measurement Statistics for Number States

Slide 5 reprises the classical versus quantum picture that we presented last time for the quadrature behavior of classical and quantum harmonic oscillators. We were a little vague, last time, about the meaning of the phasor and time-evolution plots for the quantum case, so let's try to make them precise for the case of a quantum harmonic oscillator that is in its number state $|n\rangle$. What we'd like to see is that classical physics—noiseless sinusoidal oscillation—emerges as quantum behavior in the limit of large quantum numbers. So, we'll derive the quadrature-measurement statistics when the state is $|n\rangle$ and see what happens as $n \to \infty$. Before doing so, let's note that desired classical limit behavior is already exhibited by the number state $|n\rangle$ insofar as energy-measurement statistics are concerned. Because $|n\rangle$ is an eigenket of the Hamiltonian with eigenvalue $\hbar\omega(n + 1/2)$ we know that

$$\Pr(\hat{H} \text{ measurement} = \hbar\omega(n + 1/2) \mid \text{state} = |n\rangle) = 1, \tag{7}$$

so the energy is always quantized. However, as $n \to \infty$ the $\hbar\omega$ granularity becomes imperceptibly small, compared to the energy in the state.

At this point in our development, we don't have enough theoretical machinery to fully characterize the quadrature measurement statistics. So, we will limit our attention to the mean values and variances of the the quadrature measurements. For the mean values we have that

$$\langle n|\hat{a}(t)|n\rangle = \langle n|\hat{a}|n\rangle e^{-j\omega t} = \langle n| \left(\sum_{m=1}^{\infty} \sqrt{m}\, |m-1\rangle\langle m| \right) |n\rangle e^{-j\omega t} = 0, \tag{8}$$

from which it follows that $\langle \hat{a}_1(t)\rangle = \langle \hat{a}_2(t)\rangle = 0$ when the oscillator is in a number state. Evidently, the number state *cannot* give us noiseless classical oscillation in the limit $n \to \infty$, because its mean value for both quadratures is always zero. Despite this failure, it is still worth looking into the variance of the quadrature measurements when the oscillator is in a number state. Now we find that

$$\langle n|\Delta\hat{a}_1^2(t)|n\rangle = \langle n|\hat{a}_1^2(t)|n\rangle = \langle n| \left(\frac{[\hat{a}(t)+\hat{a}^\dagger(t)]^2}{4} \right) |n\rangle \tag{9}$$

$$= \frac{\langle n|\hat{a}^2(t)|n\rangle + \langle n|\hat{a}(t)\hat{a}^\dagger(t)|n\rangle + \langle n|\hat{a}^\dagger(t)\hat{a}(t)|n\rangle + \langle n|\hat{a}^{\dagger 2}(t)|n\rangle}{4} \tag{10}$$

$$= \frac{2\langle n|\hat{a}^\dagger\hat{a}|n\rangle + 1}{4} = \frac{2n+1}{4}. \tag{11}$$

A similar calculation—left as an exercise for the reader—leads to

$$\langle n|\Delta\hat{a}_2^2(t)|n\rangle = \frac{2n+1}{4}. \tag{12}$$

Thus we see that the number state has equal uncertainties is each quadrature with an uncertainty product,

$$\langle \Delta\hat{a}_1^2(t)\rangle\langle \Delta\hat{a}_2^2(t)\rangle = \left(\frac{2n+1}{4}\right)^2 \ge \frac{1}{16}, \tag{13}$$

that equals the Heisenberg uncertainty principle limit if and only if $n = 0$. So, the zero-photon (vacuum) state $|0\rangle$ is a minimum uncertainty-product state for the quadrature components of the annihilation operator, but all the other number states have higher than minimum uncertainty products.

Slide 7 is a pictorial summary of what we have just learned. Classically, the oscillator can undergo noiseless sinusoidal oscillation, as illustrated by the phase space and time-evolution plots shown on the left-hand side of this slide. For a quantum oscillator that's in a number state $|n\rangle$, the mean value of the annihilation operator is zero, and the variances of the quadratures are equal and their product is larger (for $n \ge 1$) than that for a minimum uncertainty-product state. As a result, the phase space picture gets a donut-like shape, and the mean and mean $\pm$ one standard deviation plots in the time domain are constants, with the mean being zero. This is *not* behavior that will lead to a classical limit of noiseless sinusoidal oscillation.

Coherent States and Their Measurement Statistics

Because the classical function $ae^{-j\omega t}$ became the quantum operator $\hat{a}e^{-j\omega t}$, we might guess that the quantum states that lead to the classical limit of noiseless sinusoidal oscillation would be the eigenkets of $\hat{a}e^{-j\omega t}$. The problem here is that $\hat{a}$ is *not* Hermitian, and, in general, non-Hermitian operators do *not* have eigenkets. Nevertheless, we'll press our luck and seek such eigenkets. In particular, with α being an arbitrary complex number, we will seek a corresponding ket $|\alpha\rangle$, such that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{14}$$

If we succeed, then we'll have kets $\{|\alpha\rangle\}$ for which

$$\hat{a}(t)|\alpha\rangle = \hat{a}e^{-j\omega t}|\alpha\rangle = \alpha e^{-j\omega t}|\alpha\rangle, \tag{15}$$

thus giving us the desired sinusoidal oscillation in the mean.

The only kets that we have at our disposal now are the number kets, $\{|n\rangle\}$. These form a complete orthonormal set, so we can define

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha)|n\rangle, \tag{16}$$

and try to find coefficients $\{c_n(\alpha)\}$ such that (14) is satisfied. Using the number-ket representation of $\hat{a}$ we find that the $\{c_n(\alpha)\}$ must obey

$$\hat{a}\sum_{m=0}^{\infty} c_m(\alpha)|m\rangle = \sum_{m=1}^{\infty} c_m(\alpha)\sqrt{m}|m-1\rangle = \alpha\sum_{n=0}^{\infty} c_n(\alpha)|n\rangle. \tag{17}$$

Because the number kets are orthonormal, this equation can only be satisfied if the coefficients of $|k\rangle$, for $k = 0, 1, 2, \ldots$, are the same on both sides of the equality. More explicitly, by setting the summing index m equal to $n+1$, we get the recursion

$$\sqrt{n+1}\, c_{n+1}(\alpha) = \alpha c_n(\alpha), \tag{18}$$

whose solution is

$$c_n(\alpha) = \frac{\alpha^n}{\sqrt{n!}}\, c_0(\alpha). \tag{19}$$

Because we need $|\alpha\rangle$ to be unit length, we must enforce

$$\langle\alpha|\alpha\rangle = \left(\sum_{m=0}^{\infty} c_m^*(\alpha)\langle m|\right)\left(\sum_{n=0}^{\infty} c_n(\alpha)|n\rangle\right) = \sum_{n=0}^{\infty} |c_n(\alpha)|^2 = 1, \tag{20}$$

which becomes

$$|c_0|^2 e^{|\alpha|^2} = 1, \tag{21}$$

from (19) and the Taylor series for the exponential function. We shall take $c_0 = e^{-|\alpha|^2/2}$, giving us the $\hat{a}$ eigenkets[2]

$$|\alpha\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}}\,|n\rangle, \quad \text{for } \alpha \in \mathcal{C}, \tag{22}$$

where $\mathcal{C}$ denotes the set of complex numbers.

The $\hat{a}$ eigenkets are called *coherent states* (or Glauber coherent states, after their discoverer, Nobel Laureate Roy Glauber). They have many important properties, as we will see today and in subsequent lectures. First, however, let us emphasize that being able to find eigenkets of the non-Hermitian operator $\hat{a}$ was indeed unusual.[3] Next, let us explore some of the properties of the coherent states. Unlike the kets of an Hermitian operator, coherent states with different eigenvalues are *not* orthogonal. In particular, we find that

$$\langle\alpha|\beta\rangle = \left(\sum_{m=0}^{\infty} \frac{\alpha^{*m} e^{-|\alpha|^2/2}}{\sqrt{m!}}\,\langle m|\right)\left(\sum_{n=0}^{\infty} \frac{\beta^n e^{-|\beta|^2/2}}{\sqrt{n!}}\,|n\rangle\right) \tag{23}$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^{*n}\beta^n e^{-(|\alpha|^2+|\beta|^2|)/2}}{n!} = \exp(-|\alpha|^2/2 - |\beta|^2/2 + \alpha^*\beta). \tag{24}$$

Yet, even though the coherent states are *not* orthonormal, the do resolve the identity. In particular, we have that

$$\hat{I} = \int \frac{\mathrm{d}^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|, \tag{25}$$

where

$$\int \mathrm{d}^2\alpha \text{ is shorthand for } \int_{-\infty}^{\infty} \mathrm{d}\alpha_1 \int_{-\infty}^{\infty} \mathrm{d}\alpha_2$$

with $\alpha_1 \equiv \mathrm{Re}(\alpha)$ and $\alpha_2 \equiv \mathrm{Im}(\alpha)$. Before proceeding to the proof of this relation, we note that it means that the coherent states are *overcomplete*, i.e., they are a non-orthogonal set that resolves the identity, cf. the simple linear algebra example from $\mathcal{R}^2$ that appeared on Problem Set 1.

To prove that the coherent states resolve the identity it suffices to show that

$$\langle m|\left(\int \frac{\mathrm{d}^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|\right)|n\rangle = \delta_{mn}, \tag{26}$$

as any operator on the state space of the oscillator is completely characterized by its number-ket matrix elements, and $\langle m|\hat{I}|n\rangle = \delta_{mn}$. Bringing the $\langle m|$ and $|n\rangle$ inside the

[2]The absolute phase of c_0 does not affect the $|\alpha\rangle$-state measurement statistics for any observable, hence no generality is lost by taking c_0 to be positive real.

[3]On the homework, you will try to find eigenkets of the creation operator—kets $\{|\beta\rangle\}$ that satisfy $\hat{a}^\dagger|\beta\rangle = \beta|\beta\rangle$, for $\beta \in \mathcal{C}$—and show that they do not exist.

integral and using the number-ket representation of $|\alpha\rangle$ leads to

$$\langle m|\left(\int \frac{\mathrm{d}^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|\right)|n\rangle = \int \frac{\mathrm{d}^2\alpha}{\pi}\,\frac{\alpha^m\alpha^{*n}}{\sqrt{m!n!}}\,e^{-|\alpha|^2} = \int_0^\infty \mathrm{d}r\,r\frac{r^{m+n}e^{-r^2}}{\sqrt{m!n!}}\int_0^{2\pi}\frac{\mathrm{d}\theta}{\pi}\,e^{j(m-n)\theta}, \tag{27}$$

where (r,θ) is the polar-coordinate form of the Cartesian coordinates (α_1,α_2). Now, because

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{\pi}\,e^{j(m-n)\theta} = 2\delta_{mn}, \tag{28}$$

we need only show that

$$\int_0^\infty \mathrm{d}r\,\frac{2r^{2n+1}e^{-r^2}}{n!} = 1, \tag{29}$$

to complete our proof. That this is so follows from the change of variable $z = r^2$, so that

$$\int_0^\infty \mathrm{d}r\,\frac{2r^{2n+1}e^{-r^2}}{n!} = \int_0^\infty \mathrm{d}z\,\frac{z^n e^{-z}}{n!} = 1, \tag{30}$$

where the last equality was given in Problem Set 1.

On a future homework you will explore some consequences of (25), among them

$$\hat{a} = \int \frac{\mathrm{d}^2\alpha}{\pi}\,\alpha|\alpha\rangle\langle\alpha|, \tag{31}$$

but we will devote the rest of today's lecture to the number-measurement and quadrature-measurement statistics of the coherent states.

Number-Measurement Statistics

Suppose the oscillator is in the coherent state $|\alpha\rangle$ and we measure the number operator $\hat{N}$. The outcome will be a non-negative integer—the number of detected energy quanta (photons)—with the following probability distribution,

$$\Pr(\hat{N}\text{ measurement} = n \mid \text{state} = |\alpha\rangle) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{n!}\,e^{-|\alpha|^2}, \quad \text{for } n = 0,1,2,\ldots, \tag{32}$$

which we see is a Poisson distribution with mean $|\alpha|^2$. Unless $\alpha = 0$, in which case we have a coherent state $\hat{a}|0\rangle = 0|0\rangle$ that is also a number ket $\hat{N}|0\rangle = 0|0\rangle$, we get a number-measurement distribution that has a positive variance $|\alpha|^2$, because $|\alpha\rangle$ for $\alpha \neq 0$ is *not* an eigenket of the number operator. The signal-to-noise ratio in the number measurement,

$$\text{SNR} \equiv \frac{\langle\alpha|\hat{N}|\alpha\rangle^2}{\langle\alpha|\Delta\hat{N}^2|\alpha\rangle} = |\alpha|^2 \to \infty, \quad \text{as } |\alpha| \to \infty. \tag{33}$$

Thus the randomness in the number measurement becomes insignificantly small for a coherent state as the squared magnitude of its eigenvalue grows without bound. This is the desired classical limit behavior for the number (or energy) measurement. But, we had no problem with the classical limit for the number (or energy) measurement when we were in a number state $|n\rangle$, so the real test of the importance of coherent states will come in the next subsection, where we look at their quadrature-measurement statistics. In that case the number kets did *not* lead to the desired classical limit of noiseless sinusoidal oscillation. One final comment is in order, however, before turning to the quadrature-measurement statistics. That the coherent states are the quantum representation of classical physics is already hinted at by the Poisson distribution we have found for their number-measurement statistics: in Lecture 1 you were told that the semiclassical theory of photodetection—which uses classical electromagnetic fields plus the shot noise associated with the discreteness of the electron charge—is governed by Poisson statistics.

Quadrature-Measurement Statistics

For the quadrature-measurement statistics of the coherent state we will again limit our consideration—in this lecture—to the the behaviors of the means and variances. We already know that the mean values of the quadratures obey classical sinusoidal motion, viz.,

$$\langle\alpha|\hat{a}(t)|\alpha\rangle = \langle\alpha|\hat{a}|\alpha\rangle e^{-j\omega t} = \alpha e^{-j\omega t}, \tag{34}$$

so that

$$\langle\alpha|\hat{a}_1(t)|\alpha\rangle = \mathrm{Re}(\alpha e^{-j\omega t}) \quad \text{and} \quad \langle\alpha|\hat{a}_2(t)|\alpha\rangle = \mathrm{Im}(\alpha e^{-j\omega t}). \tag{35}$$

For the variance of the $\hat{a}_1(t)$ measurement we have that

$$\langle\alpha|\Delta\hat{a}_1^2(t)|\alpha\rangle = \langle\alpha|\hat{a}_1^2(t)|\alpha\rangle - [\mathrm{Re}(\alpha e^{-j\omega t})]^2, \tag{36}$$

where

$$\begin{aligned}
\langle\alpha|\hat{a}_1^2(t)|\alpha\rangle &= \langle\alpha|\frac{(\hat{a}e^{-j\omega t} + \hat{a}^\dagger e^{j\omega t})^2}{4}|\alpha\rangle & (37)\\
&= \frac{\langle\alpha|\hat{a}^2|\alpha\rangle e^{-2j\omega t} + \langle\alpha|\hat{a}^{\dagger 2}|\alpha\rangle e^{2j\omega t} + \langle\alpha|\hat{a}\hat{a}^\dagger|\alpha\rangle + \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle}{4} & (38)\\
&= \frac{\alpha^2 e^{-2j\omega t} + \alpha^{*2} e^{2j\omega t} + (|\alpha|^2 + 1) + |\alpha|^2}{4} & (39)\\
&= \frac{1}{4} + \left(\frac{\alpha e^{-j\omega t} + \alpha^* e^{j\omega t}}{2}\right)^2. & (40)
\end{aligned}$$

It follows that

$$\langle\alpha|\Delta\hat{a}_1^2(t)|\alpha\rangle = 1/4, \tag{41}$$

and a similar derivation shows that

$$\langle\alpha|\Delta\hat{a}_2^2(t)|\alpha\rangle = 1/4, \tag{42}$$

Thus, the coherent states have equal—and time-independent—uncertainties in each quadrature and satisfy the Heisenberg uncertainty relation,

$$\langle\Delta\hat{a}_1^2(t)\rangle\langle\Delta\hat{a}_2^2(t)\rangle \geq 1/16, \tag{43}$$

with equality.

Turning to Slide 10, we see that the coherent states have the desired classical limit behavior for the quadrature components of the oscillator. Their mean undergoes simple harmonic motion with an amplitude equal to the magnitude of the coherent-state eigenvalue, and their standard deviations remain constant. So, as $|\alpha| \to \infty$ the quadrature fluctuations become insignificant in comparison to the peak-to-peak swing of the mean value, and we get noiseless sinusoidal oscillation.

One final point is worth making in today's lecture. The zero-point energy, $\hbar\omega/2$, appearing in the Hamiltonian and its associated energy eigenvalues $\{E_n = \hbar\omega(n + 1/2) : n = 0, 1, 2, \ldots\}$ manifests itself very differently in the quadrature measurements. Consider the zero-quantum (vacuum) state $|0\rangle$. We have already noted that it is both a number state (energy eigenket) with eigenvalue 0 (energy eigenvalue $\hbar\omega/2$) and a coherent state with eigenvalue 0. Even though there are no photons in this state, quadrature measurements made on it yield classical outcomes that are zero-mean, variance-1/4 random variables. The non-zero quadrature-measurement noises—quantified by these variances—originate from the zero-point energy of the oscillator and are called *zero-point fluctuations*.

The Road Ahead

The coherent states are the quantum states that correspond to the classical field, i.e., an ideal laser produces coherent state light. However, coherent states are *not* the only possible minimum uncertainty-product states for the oscillator's quadrature components. In the next lecture we shall introduce the squeezed states, which are minimum uncertainty-product states for the quadratures that have *unequal* variances. Squeezed states are non-classical states, so, as mentioned in Lecture 1, they lead to photodetection behavior that cannot be explained by semiclassical (classical fields plus shot noise) theory.

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