Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Mid-Term Examination Solutions
Fall 2016

Problem 1 (20 points)
For each statement below, indicate whether it is True or whether it is False, and provide a brief explanation of your reasoning.

(a) (10 points) Consider a pair of single-mode electromagnetic fields, with annihilation operators $\hat{a}_A$ and $\hat{a}_B$, whose joint state $|\psi\rangle_{AB}$ is a pure state. Suppose that the $\hat{N}_A = \hat{a}_A^\dagger \hat{a}_A$ and $\hat{N}_B = \hat{a}_B^\dagger \hat{a}_B$ measurements are made on these modes and that the resulting classical outcomes, N_A and N_B, have measurement statistics which satisfy

$$\mathrm{Var}(N_A - N_B) < \mathrm{Var}(N_A) + \mathrm{Var}(N_B),$$

where $\mathrm{Var}(\cdot)$ denotes variance.

True or False: *The joint state of the $\hat{a}_A$ and $\hat{a}_B$ modes* must *be non-classical.*

This statement is *true*. The only pure-state $|\psi\rangle_{AB}$ that is classical is the two-mode coherent state, $|\psi\rangle_{AB} = |\alpha_A\rangle_A|\alpha_B\rangle_B$, and semiclassical photodetection theory gives correct measurement statistics for this state. Semiclassical photodetection theory tells us that for $|\psi\rangle_{AB} = |\alpha_A\rangle_A|\alpha_B\rangle_B$ the photon-count variances obey $\mathrm{Var}(N_A) = |\alpha_A|^2$ and $\mathrm{Var}(N_B) = |\alpha_B|^2$. Moreover, semiclassical photodetection theory also tells us that these variances are due to shot noise and that the shot noises from different photodetectors are statistically independent random variables. So, for $|\psi\rangle_{AB}$ a classical state, we have that $\mathrm{Var}(N_A - N_B) = \mathrm{Var}(N_A) + \mathrm{Var}(N_B)$. Hence for us to have $\mathrm{Var}(N_A - N_B) < \mathrm{Var}(N_A) + \mathrm{Var}(N_B)$, the joint state $|\psi\rangle_{AB}$ must be non-classical.

(b) (10 points) Consider a single-mode electromagnetic field with photon annihilation operator $\hat{a}$ whose Wigner distribution is $W(\alpha^*, \alpha)$.

True or False: *The function $F(\alpha_1) \equiv \int_{-\infty}^{\infty} \mathrm{d}\alpha_2\, W(\alpha^*, \alpha)$, where α_1 and α_2 are the real and imaginary parts of α, is non-negative for* all *values of α_1.*

This statement is *true*. To show that, we use the relation between the Wigner distribution and the Wigner characteristic function,

$$W(\alpha^*, \alpha) = \int \frac{\mathrm{d}^2\zeta}{\pi^2}\, \chi_W(\zeta^*, \zeta) e^{\zeta^*\alpha - \zeta\alpha^*},$$

plus $\zeta^*\alpha - \zeta\alpha^* = -2j\zeta_2\alpha_1 + 2j\zeta_1\alpha_2$ and get

$$\begin{aligned}
F(\alpha_1) &= \int \frac{\mathrm{d}^2\zeta}{\pi}\,\chi_W(\zeta^*,\zeta)e^{-2j\zeta_2\alpha_1}\int\frac{\mathrm{d}\alpha_2}{\pi}\,e^{2j\zeta_1\alpha_2} \\
&= \int\frac{\mathrm{d}\zeta_2}{\pi}\int \mathrm{d}\zeta_1\,\chi_W(\zeta^*,\zeta)e^{-2j\zeta_2\alpha_1}\delta(\zeta_1) \\
&= \int\frac{\mathrm{d}\zeta_2}{\pi}\,\chi_W(-\zeta_2,\zeta_2)e^{-2j\zeta_2\alpha_1} \\
&= \int\frac{\mathrm{d}\zeta_2}{\pi}\langle e^{j\zeta_2(\hat a+\hat a^\dagger)}\rangle e^{-2j\zeta_2\alpha_1} \\
&= \int\frac{\mathrm{d}\zeta_2}{\pi}\langle e^{2j\zeta_2\hat a_1}\rangle e^{-2j\zeta_2\alpha_1} = p(\alpha_1),
\end{aligned}$$

where $p(\alpha_1)$ is the probability density function (pdf) for homodyne detection of the $\hat a_1 = \mathrm{Re}(\hat a)$ quadrature to yield outcome α_1. Because pdfs must be non-negative, we have that $F(\alpha_1)$ is non-negative for all α_1.

Problem 2 (40 points)
Consider the asymmetric beam-splitter setup shown in Fig. 1. In this setup, the beam spitter is illuminated by a signal mode (with annihilation operator $\hat a_S$) and a local-oscillator (LO) mode (with annihilation operator $\hat a_{\mathrm{LO}}$). We will be interested in the output mode from that beam splitter whose annihilation operator is $\hat a_{\mathrm{out}} = \sqrt{\epsilon}\,\hat a_S + \sqrt{1-\epsilon}\,\hat a_{\mathrm{LO}}$, where $0 < \epsilon < 1$ and the $\hat a_{\mathrm{LO}}$ mode is in the coherent state $|\beta\sqrt{\epsilon/(1-\epsilon)}\rangle_{\mathrm{LO}}$.

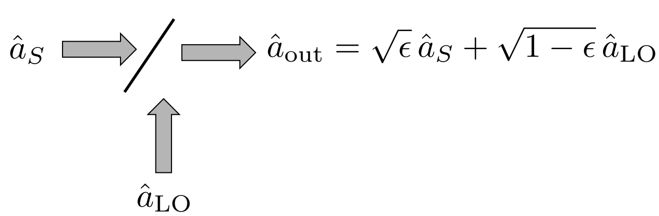

Figure 1: Asymmetric beam-splitter setup.

(a) (10 points) Suppose that the $\hat a_S$ mode is in the coherent state $|\gamma\rangle_S$.

(i) *With only a* simple *statement of justification, find the state of the $\hat a_{\mathrm{out}}$ mode.*

When a beam splitter's two input ports are illuminated by coherent states, then its two output ports are in coherent states whose eigenvalues are found by propagating the input-modes' mean values through the beam-splitter relation. So, for the case at hand, we have that the state of the $\hat{a}_{\text{out}}$ mode is the coherent state $|\sqrt{\epsilon}\,(\gamma+\beta)\rangle_{\text{out}}$.

(ii) *Use your result from (i) to find* $\rho^{(n)}_{a_{\text{out}}}(\alpha^*,\alpha) \equiv {}_{\text{out}}\langle\alpha|\hat{\rho}_{a_{\text{out}}}|\alpha\rangle_{\text{out}}$ *in the limit* $\epsilon \to 1$.

Before letting $\epsilon \to 1$, we have that

$$\rho^{(n)}_{a_{\text{out}}}(\alpha^*,\alpha) \equiv {}_{\text{out}}\langle\alpha|\hat{\rho}_{a_{\text{out}}}|\alpha\rangle_{\text{out}} = |{}_{\text{out}}\langle\alpha|\sqrt{\epsilon}\,(\gamma+\beta)\rangle_{\text{out}}|^2 = e^{-|\alpha-\sqrt{\epsilon}\,(\gamma+\beta)|^2}.$$

After we let $\epsilon \to 1$ we get $\rho^{(n)}_{a_{\text{out}}}(\alpha^*,\alpha) = e^{-|\alpha-\gamma-\beta|^2}$.

(b) (10 points) Figure 2 uses the beam-splitter setup in a photon-counting communication receiver with the following characteristics.

- The binary message b being communicated is equally likely to be 0 or 1.
- When $b = 0$, the $\hat{a}_S$ mode is in the coherent state $|-\sqrt{N_S}\rangle_S$. When $b = 1$, the $\hat{a}_S$ mode is in the coherent state $|\sqrt{N_S}\rangle_S$.
- The beam-splitter setup has $0 < \epsilon < 1$ and $\beta = \sqrt{N_S}$.
- The receiver's output is $\tilde{b} = 1$ when the $\hat{N}_{\text{out}} = \hat{a}^\dagger_{\text{out}}\hat{a}_{\text{out}}$ measurement's outcome N_{out} is non-zero. The receiver's output is $\tilde{b} = 0$ when $N_{\text{out}} = 0$.

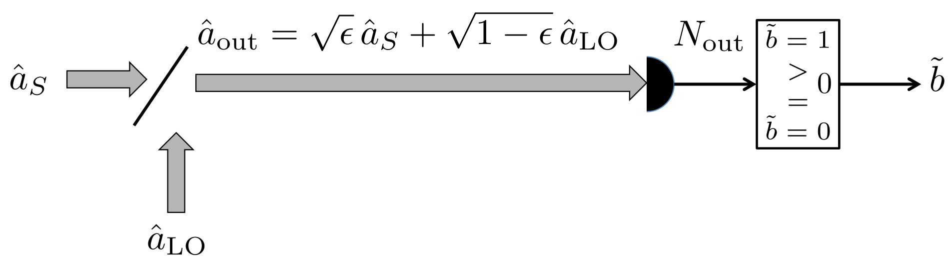

Figure 2: Photon-counting communication receiver.

(i) *Use your result from (a) to find the states that the* $\hat{a}_{\text{out}}$ *mode is in when* $b = 0$ *and* $b = 1$.

The beam-splitter's inputs are both coherent states when $b = 0$, so the result from (a) plus the states given in this part imply that the $\hat{a}_{\text{out}}$ mode is in the vacuum state $|0\rangle_{\text{out}}$ when $b = 0$. The beam-splitter's inputs are both coherent states when $b = 1$, so the results from (a) plus the states given here imply that the $\hat{a}_{\text{out}}$ mode is in the coherent state $|2\sqrt{\epsilon N_S}\rangle_{\text{out}}$ when $b = 1$.

(ii) *Use your results from (i) to find the receiver's error probability,* $\Pr(\tilde{b} \neq b)$.

We have that

$$
\begin{aligned}
\Pr(\tilde{b} \neq b) &= \Pr(\tilde{b} = 1, b = 0) + \Pr(\tilde{b} = 0, b = 1) \\
&= \Pr(b = 0)\Pr(\tilde{b} = 1 \mid b = 0) + \Pr(b = 1)\Pr(\tilde{b} = 0 \mid b = 1) \\
&= \frac{1}{2}\Pr(N_{\text{out}} > 0 \mid b = 0) + \frac{1}{2}\Pr(N_{\text{out}} = 0 \mid b = 1) \\
&= e^{-4\epsilon N_S}/2.
\end{aligned}
$$

For $\epsilon \to 1$ and $N_S \gg 1$, comparing this answer to the binary phase-shift keying results from Homework Problem 8.4(d) shows that the Fig. 3 receiver's error probability is only a factor of two higher than that of the optimum quantum receiver, and the Fig. 3 receiver's error probability is significantly lower than that of the optimum homodyne receiver.

(c) (10 points) Now, let the $\hat{a}_S$ mode be in an *arbitrary* state specified by the density operator $\hat{\rho}_S$.

(i) *Find $\chi_A^{\rho_{a_{\text{out}}}}(\zeta^*, \zeta)$, the anti-normally ordered characteristic function of the $\hat{a}_{\text{out}}$ mode. Your answer should be expressed in terms of the $\hat{a}_S$ mode's anti-normally ordered characteristic function, β, and ϵ.*

This calculation is straightforward. We have that

$$
\begin{aligned}
\chi_A^{\rho_{a_{\text{out}}}}(\zeta^*, \zeta) &= \langle e^{-\zeta^* \hat{a}_{\text{out}}} e^{\zeta \hat{a}_{\text{out}}^\dagger} \rangle \\
&= \langle e^{-\zeta^*(\sqrt{\epsilon}\,\hat{a}_S + \sqrt{1-\epsilon}\,\hat{a}_{\text{LO}})} e^{\zeta(\sqrt{\epsilon}\,\hat{a}_S^\dagger + \sqrt{1-\epsilon}\,\hat{a}_{\text{LO}}^\dagger)} \rangle \\
&= \chi_A^{\rho_{a_S}}(\sqrt{\epsilon}\,\zeta^*, \sqrt{\epsilon}\,\zeta)\chi_A^{\rho_{a_{\text{LO}}}}(\sqrt{1-\epsilon}\,\zeta^*, \sqrt{1-\epsilon}\,\zeta) \\
&= \chi_A^{\rho_{a_S}}(\sqrt{\epsilon}\,\zeta^*, \sqrt{\epsilon}\,\zeta) e^{-\zeta^*\sqrt{\epsilon}\,\beta + \zeta\sqrt{\epsilon}\,\beta^* - (1-\epsilon)|\zeta|^2},
\end{aligned}
$$

where: the first equality is the definition of $\chi_A^{\rho_{a_{\text{out}}}}$; the second used the beam splitter's input-output relation; the third used the fact that the signal and LO modes' operators commute with each other plus the definitions of $\chi_A^{\rho_{a_S}}$ and $\chi_A^{\rho_{a_{\text{LO}}}}$; and the fourth used the anti-normally ordered characteristic function of a coherent state.

(ii) *Specialize your result from (i) to the limit $\epsilon \to 1$.*

When $\epsilon \to 1$ we have

$$
\chi_A^{\rho_{a_{\text{out}}}}(\zeta^*, \zeta) = \chi_A^{\rho_{a_S}}(\zeta^*, \zeta) e^{-\zeta^*\beta + \zeta\beta^*}.
$$

(d) (10 points) *For your $\chi_A^{\rho_{a_{\text{out}}}}(\zeta^*, \zeta)$ from (c), use the operator-valued inverse transform relation,*

$$
\hat{\rho}_{a_{\text{out}}} = \int \frac{\mathrm{d}^2\zeta}{\pi}\, \chi_A^{\rho_{a_{\text{out}}}}(\zeta^*, \zeta) e^{-\zeta \hat{a}_{\text{out}}^\dagger} e^{\zeta^* \hat{a}_{\text{out}}},
$$

to obtain $\rho^{(n)}_{a_{\rm out}}(\alpha^*,\alpha) \equiv {}_{\rm out}\langle\alpha|\hat\rho_{a_{\rm out}}|\alpha\rangle_{\rm out}$ *in the* $\epsilon \to 1$ *limit. Your answer should be expressed in terms of* $\rho^{(n)}_S(\alpha^*,\alpha) \equiv {}_S\langle\alpha|\hat\rho_{a_S}|\alpha\rangle_S$*, and* β*.*

The calculation proceeds as follows.

$$\begin{aligned}
\rho^{(n)}_{a_{\rm out}}(\alpha^*,\alpha) &= {}_{\rm out}\langle\alpha|\hat\rho_{a_{\rm out}}|\alpha\rangle_{\rm out} \\
&= {}_{\rm out}\langle\alpha|\int\frac{{\rm d}^2\zeta}{\pi}\,\chi_A^{\rho_{a_{\rm out}}}(\zeta^*,\zeta)e^{-\zeta\hat a^\dagger_{\rm out}}e^{\zeta^*\hat a_{\rm out}}|\alpha\rangle_{\rm out} \\
&= \int\frac{{\rm d}^2\zeta}{\pi}\,\chi_A^{\rho_{a_{\rm out}}}(\zeta^*,\zeta)e^{-\zeta\alpha^*}e^{\zeta^*\alpha} \\
&= \int\frac{{\rm d}^2\zeta}{\pi}\,\chi_A^{\rho_{a_S}}(\zeta^*,\zeta)e^{-\zeta^*\beta+\zeta\beta^*}e^{-\zeta\alpha^*}e^{\zeta^*\alpha} \\
&= \rho^{(n)}_{a_S}(\alpha^*-\beta^*,\alpha-\beta).
\end{aligned}$$

Note that if $\hat\rho_S = |\gamma\rangle_{SS}\langle\gamma|$, where $|\gamma\rangle_S$ is a coherent state, the result just obtained implies that $\hat\rho_{\rm out} = |\gamma+\beta\rangle_{\rm out\,out}\langle\gamma+\beta|$, i.e., the $\hat a_{\rm out}$ mode is in the coherent state $|\gamma+\beta\rangle_{\rm out}$, as found more easily in (a). What (d) has shown is that the Fig. 1 setup with $\epsilon \to 1$ performs a mean-field translation by β on an *arbitrary* signal-mode input state.

Problem 3 (40 points)
The system shown in Fig. 3 is a quantum non-demolition (QND) setup for measuring the photon number of an optical mode with annihilation operator $\hat a$. The cross-Kerr-effect box has the following input-output relation:

$$\hat c = e^{j\kappa\hat a^\dagger\hat a}\hat b$$
$$\hat d = e^{j\kappa\hat b^\dagger\hat b}\hat a,$$

where $\kappa > 0$ is a constant. The homodyne detector is set up to measure the $\hat c_2 = {\rm Im}(\hat c)$ observable.

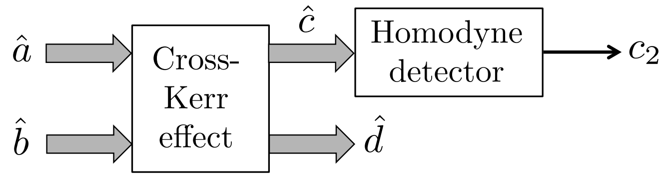

Figure 3: Quantum non-demolition detection setup.

(a) (10 points) *Evaluate the number-ket matrix elements,*

$$ {}_b\langle n_b|_a\langle n_a|e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b}e^{j\kappa\hat{b}^\dagger\hat{b}}\hat{a}|m_a\rangle_a|m_b\rangle_b $$

and

$$ {}_b\langle n_b|_a\langle n_a|e^{j\kappa\hat{b}^\dagger\hat{b}}\hat{a}e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b}|m_a\rangle_a|m_b\rangle_b. $$

Let's start with

$$ \begin{aligned} e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b}e^{j\kappa\hat{b}^\dagger\hat{b}}\hat{a}|m_a\rangle_a|m_b\rangle_b &= (e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{a}|m_a\rangle_a)(\hat{b}e^{j\kappa\hat{b}^\dagger\hat{b}}|m_b\rangle_b) \\ &= (e^{j\kappa(m_a-1)}\sqrt{m_a}\,|m_a-1\rangle_a)(\sqrt{m_b}\,e^{j\kappa m_b}|m_b-1\rangle_b), \end{aligned} $$

and

$$ \begin{aligned} e^{j\kappa\hat{b}^\dagger\hat{b}}\hat{a}e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b}|m_a\rangle_a|m_b\rangle_b &= (\hat{a}e^{j\kappa\hat{a}^\dagger\hat{a}}|m_a\rangle_a)(e^{j\kappa\hat{b}^\dagger\hat{b}}\hat{b}|m_b\rangle_b) \\ &= (\sqrt{m_a}\,e^{j\kappa m_a}|m_a-1\rangle_a)(e^{j\kappa(m_b-1)}\sqrt{m_b}\,|m_b-1\rangle_b), \end{aligned} $$

We now get the desired matrix elements:

$$ {}_b\langle n_b|_a\langle n_a|e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b}e^{j\kappa\hat{b}^\dagger\hat{b}}\hat{a}|m_a\rangle_a|m_b\rangle_b = (e^{j\kappa(m_a-1)}\sqrt{m_a}\,\delta_{n_a,m_a-1})(\sqrt{m_b}\,e^{j\kappa m_b}\delta_{n_b,m_b-1}), $$

and

$$ {}_b\langle n_b|_a\langle n_a|e^{j\kappa\hat{b}^\dagger\hat{b}}\hat{a}e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b}|m_a\rangle_a|m_b\rangle_b = (\sqrt{m_a}\,e^{j\kappa m_a}\delta_{n_a,m_a-1})(e^{j\kappa(m_b-1)}\sqrt{m_b}\,\delta_{n_b,m_b-1}), $$

where

$$ \delta_{j,k} = \begin{cases} 1, & \text{for } j = k \\ 0, & \text{for } j \neq k. \end{cases} $$

Because these matrix elements determine the operators $\hat{c}\hat{d}$ and $\hat{d}\hat{c}$, respectively, *and* because they have the same values, we have that $[\hat{c},\hat{d}] = 0$, i.e., the $\hat{c}$ and $\hat{d}$ annihilation operators commute. More generally, it can be shown that $[\hat{c},\hat{c}^\dagger] = [\hat{d},\hat{d}^\dagger] = 1$, and $[\hat{c},\hat{d}^\dagger] = 0$. Thus the cross-Kerr effect box preserves commutator operators, hence no noise modes need to be included in its input-output relation.

(b) (10 points) *Assume that the $\hat{a}$ mode is in the number state $|m_a\rangle_a$. Let N_d be the outcome of the $\hat{N}_d = \hat{d}^\dagger\hat{d}$ measurement. Find the probability mass function* $\Pr(N_d = n)$. **Hint**: *You do* not *need to know the state of the $\hat{b}$ mode.*

We have that $\hat{N}_d = \hat{d}^\dagger\hat{d} = \hat{a}^\dagger e^{-j\kappa\hat{b}^\dagger\hat{b}}e^{j\kappa\hat{b}^\dagger\hat{b}}\hat{a} = \hat{a}^\dagger\hat{a}$. So, N_d can be interpreted as the outcome of the $\hat{N}_a = \hat{a}^\dagger\hat{a}$ measurement. Because we are told that the $\hat{a}$ mode is in the number state $|m_a\rangle_a$, we have that

$$ \Pr(N_d = n) = |{}_a\langle n|m_a\rangle_a|^2 = \delta_{n,m_a}. $$

The equivalence of the $\hat{N}_d$ and $\hat{N}_a$ measurements shows that the cross-Kerr effect box does *not* disturb the $\hat{a}$ mode's photon-counting statistics. In particular, if the $\hat{a}$ mode is in a number state, then the $\hat{d}$ mode will be in that same number state.

(c) (10 points) *Assume that the $\hat{a}$ mode is in the number state $|m_a\rangle_a$ and the $\hat{b}$ mode is in the coherent state $|\sqrt{N_b}\rangle_b$. Find $\langle \hat{c}_2\rangle$ and $\langle \Delta\hat{c}_2^2\rangle$, the mean and variance of the $\hat{c}_2$ measurement.*

For the mean of $\hat{c}_2$ we have that

$$\begin{aligned}
\langle \hat{c}_2\rangle &= \left\langle \left(\frac{\hat{c}-\hat{c}^\dagger}{2j}\right)\right\rangle \\
&= {}_a\langle m_a|_b\langle\sqrt{N_b}|\frac{e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b}}{2j}|\sqrt{N_b}\rangle_b|m_a\rangle_a - {}_a\langle m_a|_b\langle\sqrt{N_b}|\frac{\hat{b}^\dagger e^{-j\kappa\hat{a}^\dagger\hat{a}}}{2j}|\sqrt{N_b}\rangle_b|m_a\rangle_a \\
&= \frac{e^{j\kappa m_a}\sqrt{N_b} - \sqrt{N_b}\,e^{-j\kappa m_a}}{2j} = \sqrt{N_b}\,\sin(\kappa m_a).
\end{aligned}$$

We'll get the variance from $\langle \Delta\hat{c}_2^2\rangle = \langle \hat{c}_2^2\rangle - \langle \hat{c}_2\rangle^2$ once we've found the mean-square via

$$\begin{aligned}
\langle \hat{c}_2^2\rangle &= \left\langle \left(\frac{\hat{c}-\hat{c}^\dagger}{2j}\right)^2\right\rangle \\
&= \left\langle \frac{(e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b})^2 + (\hat{b}^\dagger e^{-j\kappa\hat{a}^\dagger\hat{a}})^2 - \hat{b}^\dagger e^{-j\kappa\hat{a}^\dagger\hat{a}}e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b} - e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b}\hat{b}^\dagger e^{-j\kappa\hat{a}^\dagger\hat{a}}}{-4}\right\rangle \\
&= \frac{e^{2j\kappa m_a}N_b + N_b e^{-2j\kappa m_a} - 2N_b - 1}{-4} \\
&= N_b\,\frac{1-\cos(2\kappa m_a)}{2} + 1/4 \\
&= N_b\sin^2(\kappa m_a) + 1/4.
\end{aligned}$$

It follows that $\langle \Delta\hat{c}_2^2\rangle = 1/4$.

(d) (10 points) *Assume that the states of the $\hat{a}$ and $\hat{b}$ modes are as given in (c), and that $\kappa m_a \ll 1$. Let c_2 denote the outcome of the $\hat{c}_2$ measurement and define $\tilde{N}_a = c_2/\sqrt{N_b}\,\kappa$ to be the QND estimate of the $\hat{a}$ mode's photon number. Find the mean-squared error of this estimate, i.e., $\langle(\tilde{N}_a - m_a)^2\rangle$.*

This part is really easy. From (c) we find that

$$\langle \tilde{N}_a\rangle = \langle c_2\rangle/\sqrt{N_b}\,\kappa = \langle \hat{c}_2\rangle/\sqrt{N_b}\,\kappa = \sin(\kappa m_a)/\kappa \approx m_a, \text{ because } \kappa m_a \ll 1.$$

Thus, for the mean-squared error when $\kappa m_a \ll 1$ we get

$$\langle (\tilde{N}_a - m_a)^2 \rangle \approx \langle \Delta \tilde{N}_a^2 \rangle = \langle \Delta \hat{c}_2^2 \rangle / N_b \kappa^2 \approx 1/4 N_b \kappa^2.$$

Thus, when $N_b \kappa^2 \gg 1$ the QND setup's output $\tilde{N}_a$ is a very accurate estimate of the $\hat{a}$ mode's photon number.