Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 1
Fall 2016

Issued: Thursday, September 8, 2016 **Due:** Thursday, September 15, 2016

Reading: For probability review: Chapter 3 of J. H. Shapiro, *Optical Progagation, Detection, and Communication*,
For linear algebra review: Section 2.1 of M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*.

Problem 1.1

Here we shall verify the elementary properties of the 1-D Gaussian probability density function (pdf)

$$p_x(X) = \frac{e^{-(X-m)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}, \quad \text{for } -\infty < X < \infty.$$

(a) By converting from rectangular to polar coordinates, using $X - m = R\cos(\Phi)$ and $Y - m = R\sin(\Phi)$, show that

$$\left(\int_{-\infty}^{\infty} dX\, e^{-(X-m)^2/2\sigma^2}\right)^2 = \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dY\, e^{-(X-m)^2/2\sigma^2-(Y-m)^2/2\sigma^2} = 2\pi\sigma^2,$$

thus verifying the normalization constant for the Gaussian pdf.

(b) By completing the square in the exponent within the integrand,

$$\int_{-\infty}^{\infty} dX\, \frac{e^{jvX-(X-m)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}},$$

verify that

$$M_x(jv) = e^{jvm-v^2\sigma^2/2},$$

is the characteristic function associated with the Gaussian pdf.

(c) Differentiate $M_x(jv)$ to verify that $E(x) = m$; differentiate once more to verify that $\text{var}(x) = \sigma^2$.

Problem 1.2

Here we shall verify the elementary properties of the Poisson probability mass function (pmf),

$$P_x(n) = \frac{m^n}{n!}e^{-m}, \quad \text{for } n = 0, 1, 2, \ldots, \text{ and } m \geq 0.$$

(a) Use the power series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

to verify that the Poisson pmf is properly normalized.

(b) Use the power series for e^z to verify that

$$M_x(jv) = \exp[m(e^{jv} - 1)].$$

is the characteristic function associated with the Poisson pmf.

(c) Differentiate $M_x(jv)$ to verify that $E(x) = m$; differentiate once more to verify that $\text{var}(x) = m$.

Problem 1.3
Let x be a Rayleigh random variable, i.e., x has pdf

$$p_x(X) = \begin{cases} \dfrac{X}{\sigma^2} e^{-X^2/2\sigma^2}, & \text{for } X \geq 0 \\ 0, & \text{otherwise,} \end{cases}$$

and let $y = x^2$.

(a) Find $p_y(Y)$, the pdf of y.

(b) Find m_y and σ_y^2, the mean and variance of the random variable y.

Problem 1.4
Let x and y be statistically independent, identically distributed, zero-mean, variance σ^2, Gaussian random variables, i.e., the joint pdf for x and y is,

$$p_{x,y}(X, Y) = \frac{e^{-X^2/2\sigma^2 - Y^2/2\sigma^2}}{2\pi\sigma^2}.$$

Suppose we regard (x, y) as the Cartesian coordinates of a point in the plane, and let (r, ϕ) be the polar-coordinate representation of this point, viz., $x = r\cos(\phi)$ and $y = r\sin(\phi)$ for $r \geq 0$ and $0 \leq \phi < 2\pi$

(a) Find $p_{r,\phi}(R, \Phi)$, the joint pdf of r and ϕ.

(b) Find the marginal pdfs, $p_r(R)$ and $p_\phi(\Phi)$, of these random variables, and prove that r and ϕ are statistically independent random variables.

Problem 1.5
Let N, x be joint random variables. Suppose that x is exponentially distributed with mean m, i.e.,

$$p_x(X) = \begin{cases} \dfrac{e^{-X/m}}{m}, & \text{for } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

is the pdf of x. Also suppose that, given $x = X$, N is Poisson distributed with mean value X, i.e., the conditional pmf of N is,

$$P_{N|x}(\, n \mid x = X) = \frac{X^n}{n!} e^{-X}, \quad \text{for } n = 0, 1, 2, \ldots$$

(a) Use the integral formula,

$$\int_0^\infty dZ Z^n e^{-Z} = n!, \quad \text{for } n = 0, 1, 2, \ldots,$$

(where $0! = 1$) to find $P_N(n)$, the unconditional pmf of N.

(b) Find $M_N(jv)$, the characteristic function associated with your unconditional pmf from (a).

(c) Find $E(N)$ and $\text{var}(N)$, the unconditional mean and variance of N, by differentiating your characteristic function from (b).

Problem 1.6
Let x, y be jointly Gaussian random variables with zero-means $m_x = m_y = 0$, identical variances $\sigma_x^2 = \sigma_y^2 = \sigma^2$, and nonzero correlation coefficient ρ. Let w, z be two new random variables obtained from x, y by the following transformation,

$$\begin{aligned} w &= x\cos(\theta) + y\sin(\theta) \\ z &= -x\sin(\theta) + y\cos(\theta), \end{aligned}$$

for θ a deterministic angle satisfying $0 < \theta < \pi/2$.

(a) Show that this transformation is a rotation in the plane, i.e., (w, z) are obtained from (x, y) by rotation through angle θ

(b) Find $p_{w,z}(W, Z)$ the joint pdf of w and z.

(c) Find a θ value such that w and z are statistically independent.

Problem 1.7

Here we shall examine some of the eigenvalue/eigenvector properties of an Hermitian matrix. Let $\mathbf{x}$ be an N-D column vector of complex numbers whose nth element is x_n, let A be an $N \times N$ matrix of complex numbers whose ijth element is a_{ij}, and let † denote conjugate transpose so that $\mathbf{x}^\dagger = \begin{bmatrix} x_1^* & x_2^* & \cdots & x_N^* \end{bmatrix}$ and $A^\dagger$ is an $N \times N$ matrix whose ijth element is a_{ji}^*.

(a) Find the adjoint of A, i.e., the matrix B which satisfies $(B\mathbf{y})^\dagger \mathbf{x} = \mathbf{y}^\dagger (A\mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{C}^N$, where $\mathcal{C}^N$ is the space of N-D vectors with complex-valued elements. If $B = A$, for a particular matrix A, we say that A is self-adjoint, or Hermitian. Assume that A is Hermitian for parts (b)–(d)

(b) Let A have eigenvalues $\{\, \mu_n : 1 \le n \le N \,\}$ and normalized eigenvectors $\{\, \boldsymbol{\phi}_n : 1 \le n \le N \,\}$ obeying

$$\begin{aligned} A\boldsymbol{\phi}_n &= \mu_n \boldsymbol{\phi}_n, \quad \text{for } 1 \le n \le N. \\ \boldsymbol{\phi}_n^\dagger \boldsymbol{\phi}_n &= 1, \quad \text{for } 1 \le n \le N. \end{aligned}$$

Show that μ_n is real valued for $1 \le n \le N$.

(c) Show that if $\mu_n \ne \mu_m$ then $\boldsymbol{\phi}_n^\dagger \boldsymbol{\phi}_m = 0$, i.e., eigenvectors associated with distinct eigenvalues are orthogonal.

(d) Suppose there are two linearly independent eigenvectors, $\boldsymbol{\phi}$ and $\boldsymbol{\phi}'$ which have the same eigenvalue, μ. Show that two orthogonal vectors, $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ can be constructed satisfying,

$$\begin{aligned} A\boldsymbol{\theta} &= \mu\boldsymbol{\theta}, \\ A\boldsymbol{\theta}' &= \mu\boldsymbol{\theta}', \\ \boldsymbol{\theta}^\dagger \boldsymbol{\theta}' &= 0. \end{aligned}$$

(e) Because of the results of parts (c) and (d), we can assume that $\{\, \boldsymbol{\phi}_n : 1 \le n \le N \,\}$ is a complete orthornormal (CON) set of vectors on $\mathcal{C}^N$, i.e.,

$$\boldsymbol{\phi}_n^\dagger \boldsymbol{\phi}_m = \begin{cases} 1, & \text{for } n = m, \\ 0, & \text{for } n \ne m. \end{cases}$$

Let I_N be the identity matrix on this space. Show that

$$I_N = \sum_{n=1}^{N} \boldsymbol{\phi}_n \boldsymbol{\phi}_n^\dagger.$$

Show that

$$A = \sum_{n=1}^{N} \mu_n \boldsymbol{\phi}_n \boldsymbol{\phi}_n^\dagger.$$

Problem 1.8
Here we introduce the notion of overcompleteness. Consider 2-D real Euclidean space, $\mathcal{R}^2$, i.e., the space of 2-D column vectors $\mathbf{x}$ where $\mathbf{x}^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$, with x_1 and x_2 being real numbers. Define three vectors as follows:

$$\mathbf{x}_1 = \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix}.$$

(a) Make a labeled sketch of these three vectors on an (x_1, x_2) plane, and find $\mathbf{x}_n^T \mathbf{x}_m$ for $1 \le n, m \le 3$. Are these three vectors normalized (unit length)? Are they orthogonal?

(b) Show that any two of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ form a basis for the space $\mathcal{R}^2$, i.e., any $\mathbf{y} \in \mathcal{R}^2$ can be expressed as

$$\mathbf{y} = a\mathbf{x}_1 + a'\,\mathbf{x}_2 = b\mathbf{x}_1 + b'\,\mathbf{x}_3 = c\mathbf{x}_2 + c'\mathbf{x}_3,$$

for appropriate choices of the (real-valued) coefficients $\{a, a', b, b', c, c'\}$.

(c) Show that the 2×2 identity matrix, I_2, can be expressed as

$$I_2 = \frac{2}{3} \sum_{n=1}^{3} \mathbf{x}_n \mathbf{x}_n^T.$$

Use this result to prove that for any $\mathbf{x} \in \mathcal{R}^2$ that

$$\mathbf{x} = \frac{2}{3} \sum_{n=1}^{3} (\mathbf{x}_n^T \mathbf{x}) \mathbf{x}_n.$$

Comment: Let $\mathbf{e}_1^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\mathbf{e}_2^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$ be the standard basis of $\mathcal{R}^2$. They are a complete orthornormal set of vectors on $\mathcal{R}^2$, hence

$$I_2 = \sum_{n=1}^{2} \mathbf{e}_n \mathbf{e}_n^T,$$

and the standard representation for $\mathbf{x} \in \mathcal{R}^2$ can be expressed as

$$\mathbf{x} = \sum_{n=1}^{2} (\mathbf{e}_n^T \mathbf{x}) \mathbf{e}_n.$$

We say that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ form an overcomplete basis for $\mathcal{R}^2$ because any two of them is enough to represent an arbitrary vector in this space, but all three taken together resolve the identity [their outer-product-sum times a scale factor equals the identity matrix, as shown in part (c)] hence the expansion coefficients needed to represent an arbitrary vector in this overcomplete basis can be found via projection [as shown in part(c)].

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