Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: DISCRETE-TIME SIGNAL PROCESSING
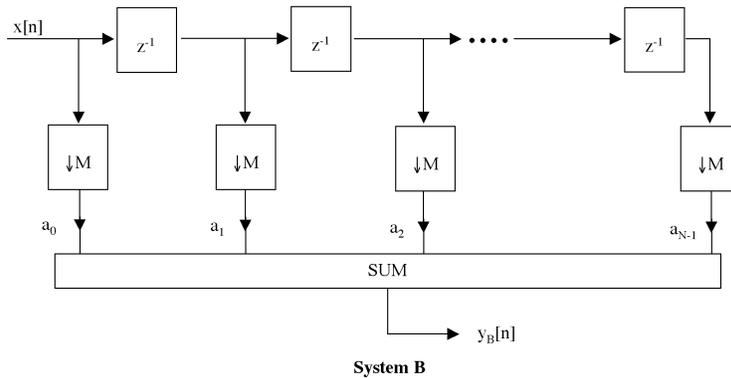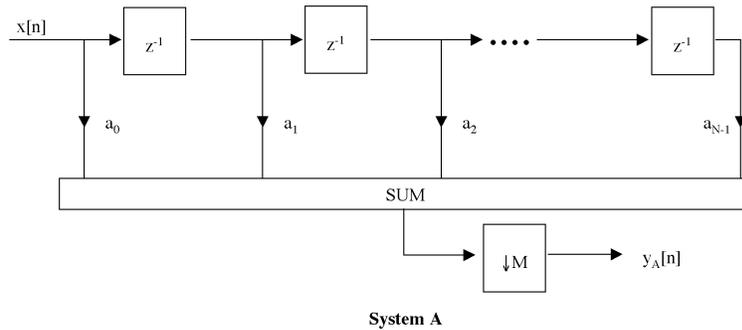
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Lecture 10
Multirate Systems and Polyphase Structures

Reading: Section 4.7 in Oppenheim, Schafer & Buck (OSB).

Consider the two systems depicted below.

System A

System B

The following questions can be posed:

(1) Do the two systems have the same input-output relationship? In other words, are $y_A[n]$ and $y_B[n]$ identical?

(2) If $x[n]$ is clocked at 1 sample/second, how many multiplications per second are required in System A?

(3) How many in System B?

The answers

(1) Yes

(2) N multiplies per second

(3) N/M multiplies per second

illustrate the point that there is often some efficiency which can be exploited when dealing with multirate systems. This should seem intuitively possible for many multirate systems; if some samples are thrown away in a compressor, for instance, multiplications which contribute to them shouldn't need to be computed.

As a prelude to further discussion, recall that an expander-by-L

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_e[n]$$

has its input and output related by

$$x_e[n] = \begin{cases} x[n/L], & n/L \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases},$$

and in the frequency domain,

$$X_e(e^{j\omega}) = X(e^{j\omega L}).$$

This relationship is depicted for $L = 2$ in OSB Figure 4.25.

Likewise, a compressor-by-M

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n]$$

has its input and output related by

$$x_d[n] = x[nM],$$

and in the frequency domain,

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega - 2\pi k}{M}\right)}\right).$$

This relationship is depicted for $M = 3$ in OSB Figure 4.22.

The efficiencies in System B on page 1 stem from using the noble identities, discussed in pp. 179-180 of OSB. The downsampling identity is illustrated in OSB Figure 4.30.

According to this figure, moving an LTI filter which is at the output of a compressor-by-M to the compressor's input expands the filter's response by M. Note that the converse is not always true; moving an LTI filter H which is at the input of a compressor-by-M to its output compresses the filter's response by M only if the LTI filter can be represented as $H(z^M)$ using integer powers of z^M.

The downsampling identity can be intuitively justified by making the following observation. When a signal $x[n]$ is processed as in OSB Figure 4.30(b) by filtering with some $H(z^M)$ (which implies $h[n] = 0 \ \forall n \neq kM, k \in \mathbb{Z}$) and compressing by a factor of M, each output sample $y[n]$ depends only on every Mth input sample. It therefore stands to reason that an equivalent input-output relationship can be achieved by first compressing, then filtering. The point is illustrated below.
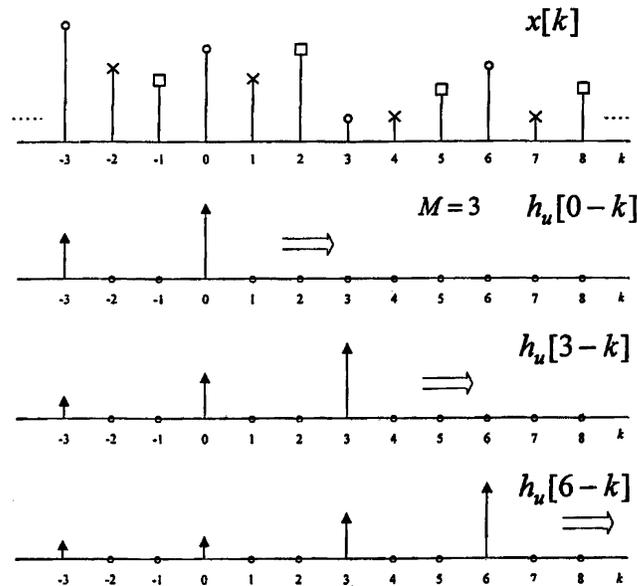
The question is then raised of how to use all this to efficiently implement LTI filters in multirate systems. It should seem reasonable to try to take advantage of the noble identities in some way, since they have the convenient property of being able to move certain LTI filters from the high-rate side to the low-rate side of compressors and expanders without affecting the overall system response.

Consider an LTI filter with impulse response $h[n]$ and the identity system depicted in OSB Figure 4.33, into which we put (and out of which we again see) the signal $h[n]$. The signals $e_k[n]$ are what we will call the kth polyphase components of $h[n]$. This decomposition is illustrated for an example $h[n]$ and $M = 3$.
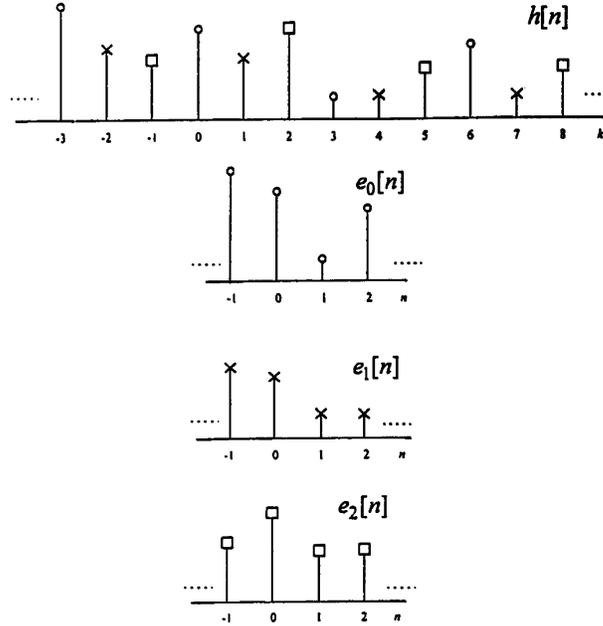
Given the polyphase components $e_k[n]$, how might the original filter $h[n]$ be implemented? One way to do this is to design a filter structure which takes expanded-by-M versions of the components $e_k[n]$ and re-aligns them so that the overall impulse response is $h[n]$. The structure in OSB Figure 4.34 accomplishes this. Its transfer function is identical to that of our original filter $h[n]$. When the structure is followed by a compressor-by-M (a common arrangement in downsampling systems), the compressor can be distributed backwards through the summation to give the system in OSB Figure 4.36.

Because each filter is represented as $E_k(z^M)$, the downsampling noble identity can be applied to each branch of the system. The filters therefore "swap" with the compressors-by-M, and all multiplications are now on the slow-rate side of the compressors, as depicted in OSB Figure 4.37. Note that the systems in OSB Figures 4.36 and 4.37 have the same input-output relationship.

Exactly what have we gained by these manipulations? Assume that the input signals to both systems are clocked in at 1 sample/second. Further assume that each filter in OSB Figure 4.36 uses N/M multipliers, as is the case when $h[n]$ is of length N and multipliers-by-0 are not counted as multipliers. A brief tour into dimensional analysis then gives

$$\frac{N/M \ \frac{\text{multiplies}}{\text{input sample}}}{\text{filter } E_k} \cdot M \text{ filters } E_k \cdot \frac{1 \text{ input sample}}{\text{second}} = N \ \frac{\text{multiplies}}{\text{second}}$$

for OSB Figure 4.36 and

$$\frac{N/M \ \dfrac{\text{multiplies}}{\text{output sample}}}{\text{filter } E_k} \cdot M \text{ filters } E_k \cdot \frac{1 \text{ output sample}}{M \text{ input samples}} \cdot \frac{1 \text{ input sample}}{\text{second}} = N/M \ \frac{\text{multiplies}}{\text{second}}$$

for OSB Figure 4.37.

As a final thought, note that the application of the downsampling identity which takes us from OSB Figure 4.36 to OSB Figure 4.37 is possible for filters $H(z^R)$, where R is an integer multiple of M. The following structure illustrates this.