MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.341 DISCRETE-TIME SIGNAL PROCESSING
Fall 2004

MIDTERM EXAM
Tuesday, November 9, 2004

NAME: Solutions

Problem	Grade	Points	Grader
1 (a)			
1 (b)			
1 (c)			
1 (d)			
2 (a)			
2 (b)			
2 (c)			
3 (a)			
3 (b)			
4 (a)			
4 (b)			
4 (c)			
Total			

Problem 1 (28%)

In figure 1-1 are two systems consisting of a compressor and an expander.

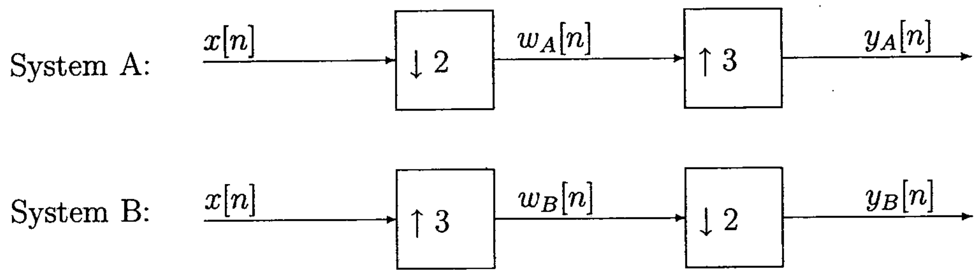


Figure 1-1:

(a) (4%) For $x[n]$ as shown in figure 1-2 sketch $y_A[n]$ and $y_B[n]$ (assume $x[n] = 0$ outside the interval shown).

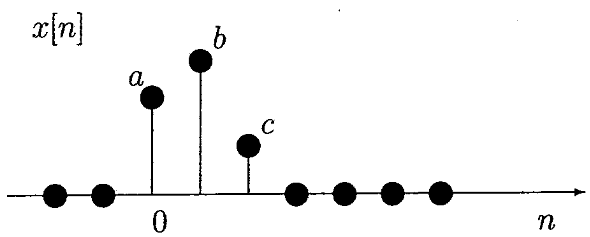
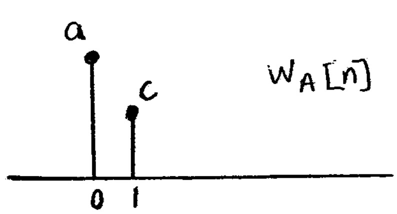
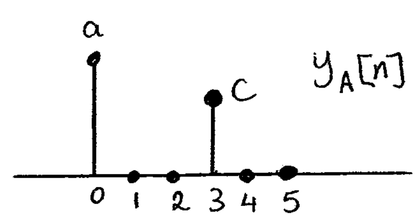
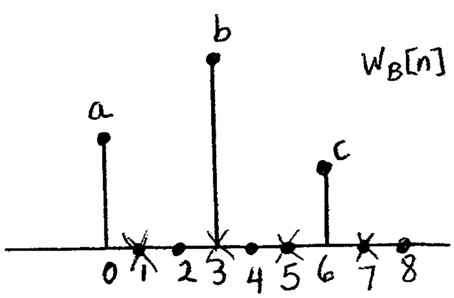
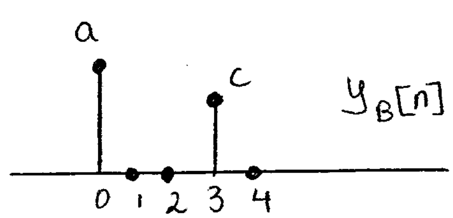


Figure 1-2:









(b) (6%) For $X(e^{j\omega})$ as shown in figure 1-3, sketch $W_a(e^{j\omega})$ and $Y_A(e^{j\omega})$.

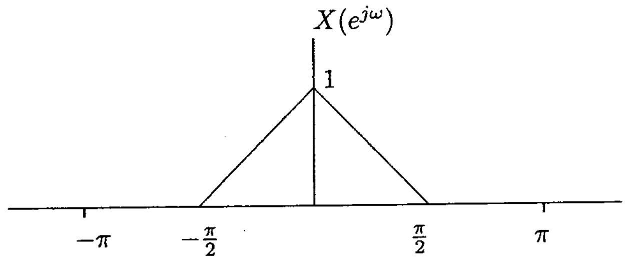
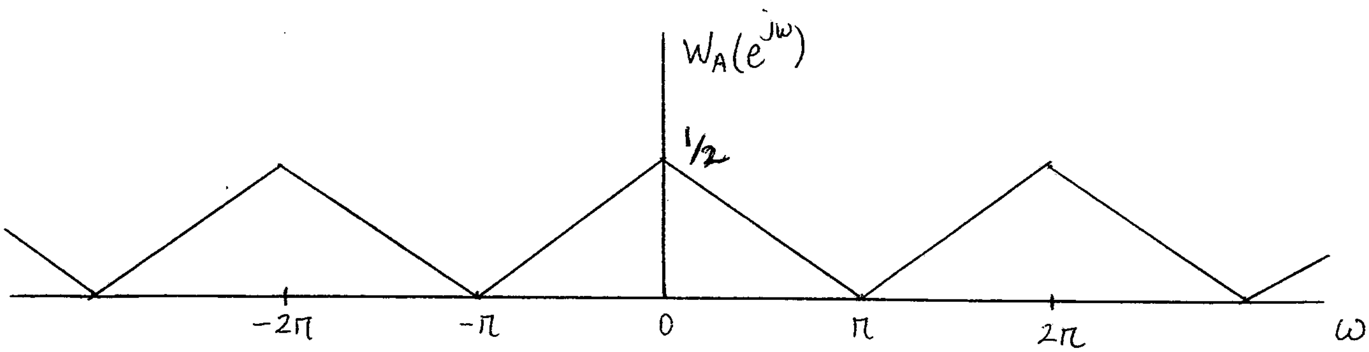
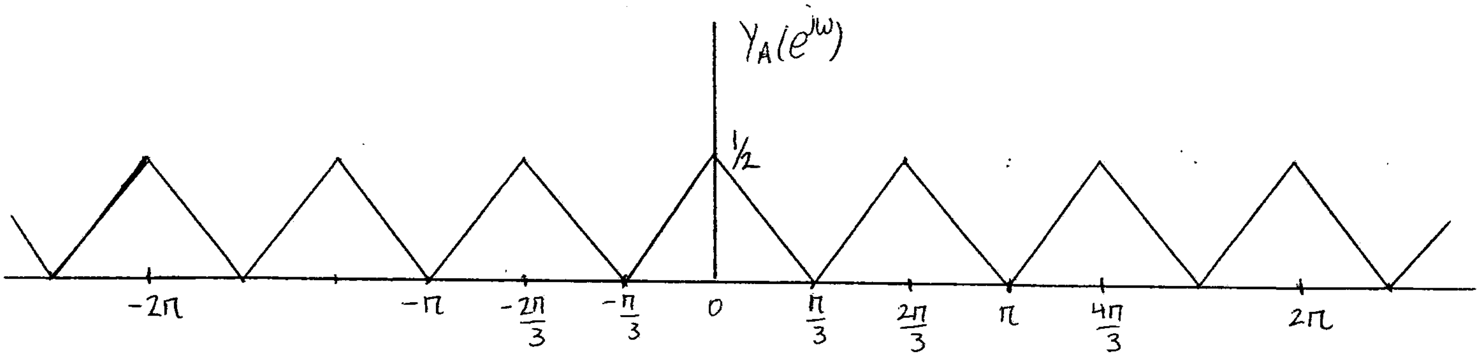


Figure 1-3:





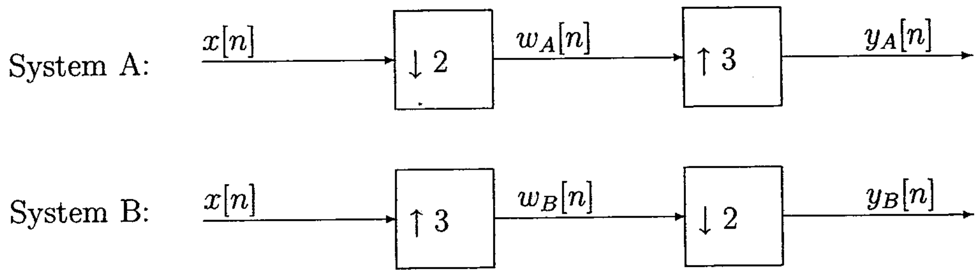


Figure 1-1 repeated for your convenience.

(c) (8%) $X(e^{j\omega})$ denotes the Fourier transform for an arbitrary $x[n]$. Express $Y_B(e^{j\omega})$ in terms of $X(e^{j\omega})$.Your answer should be in the form of an equation, not a sketch for a specific Fourier transform.

$$W_B(e^{j\omega}) = X(e^{j3\omega}) \quad , \quad Y_B(e^{j\omega}) = \frac{1}{2}\left[W_B(e^{j\frac{\omega}{2}}) + W_B\left(e^{j\left(\frac{\omega}{2}-\pi\right)}\right)\right]$$

$$Y_B(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\frac{3\omega}{2}}) + X\left(e^{j3\left(\frac{\omega}{2}-\pi\right)}\right)\right] = \frac{1}{2}\left[X(e^{j\frac{3\omega}{2}}) + X(e^{j\frac{3\omega}{2}-3\pi})\right]$$

$$Y_B(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\frac{3\omega}{2}}) + X(e^{j\frac{3\omega}{2}-\pi})\right]$$

(d) (10%) For any arbitrary $x[n]$, will $y_A[n] = y_B[n]$? If your answer is yes, algebraically justify your answer. If your answer is no, clearly explain or give a counterexample.

yes.

$$Y_A(e^{j\omega}) = W_A(e^{j3\omega}) \quad , \quad W_A(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\frac{\omega}{2}}) + X\left(e^{j\left(\frac{\omega}{2}-\pi\right)}\right)\right]$$

$$Y_A(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\frac{3\omega}{2}}) + X\left(e^{j\left(\frac{3\omega}{2}-\pi\right)}\right)\right] \quad , \quad \therefore \; Y_A(e^{j\omega}) = Y_B(e^{j\omega})$$

Therefore $y_A[n] = y_B[n]$ for any arbitrary $x[n]$.

Additional Information: this result holds whenever the rate of expansion and compression are coprime.

Problem 2 (28%)

Consider the system in Figure 2-1.

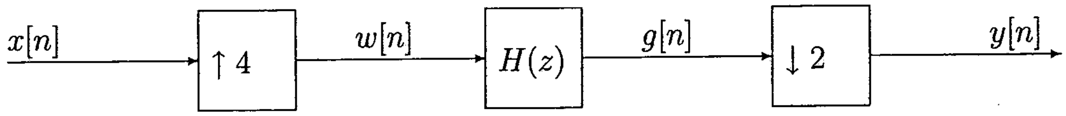


Figure 2-1:

We would like to implement the overall system (i.e. from $x[n]$ to $y[n]$ as efficiently as possible in terms of the number of multiplies per output sample, the total number of compressor/expander blocks, and the total number of delay elements.

(a) (8%) If $H(z) = \dfrac{1 - cz^{-1}}{1 - az^{-1}}$, where a and c are arbitrary and the system must be implemented exactly as shown in figure 2-1, with $H(z)$ in direct form, determine the required number of multiplies for each value of $y[n]$.

We have 2 multiplies for each value of $g[n]$. Then, going through the decimator $\boxed{\downarrow 2}$ we get 4 multiplies for each value of $y[n]$. Since we have one value of $y[n]$ for every 2 values of $g[n]$.

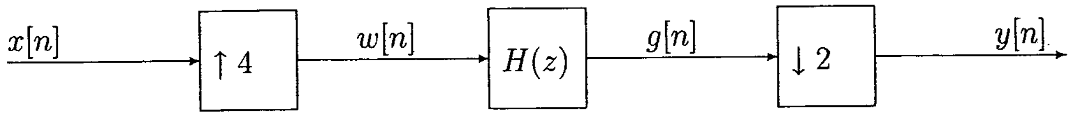
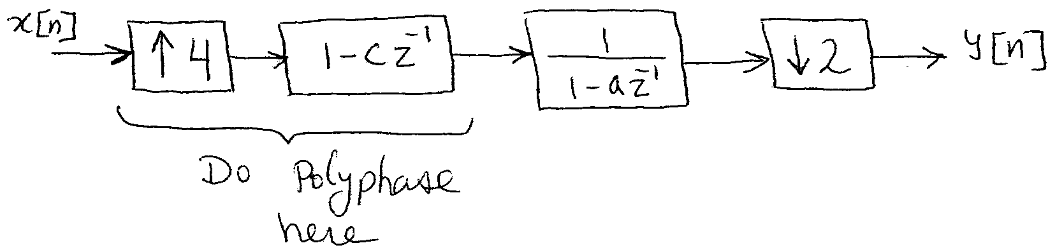
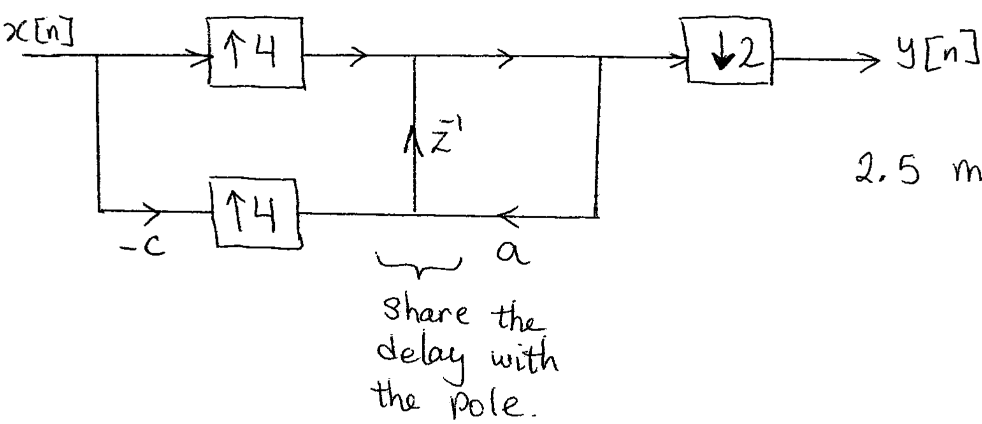


Figure 2-1 repeated for your convenience

For parts (b) and (c) assume that the cost of each multiply per output sample $y[n]$ is 10, the cost of each compressor or expander block is 5, and the cost of each delay is 2.

(b) (10%) If $H(z) = \dfrac{1 - cz^{-1}}{1 - az^{-1}}$, determine the flowgraph or block diagram of the overall system which minimizes the implementation cost of the overall system in Figure 2-1, and the total cost of the implementation.



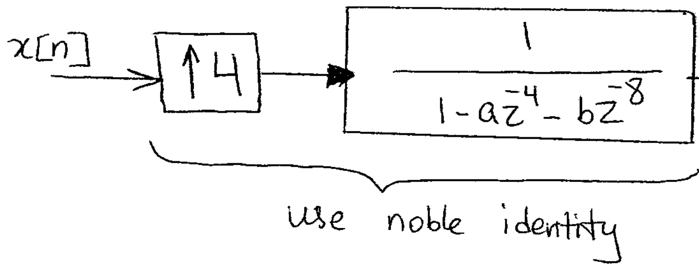


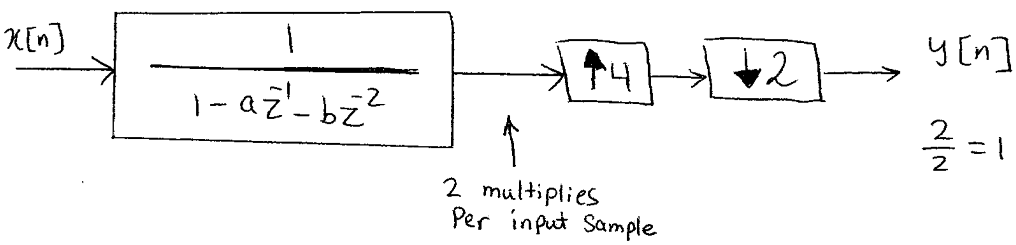
2.5 multiplies/output sample

Total Cost: $10 \times 2.5 + 3 \times 5 + 2 = 42$

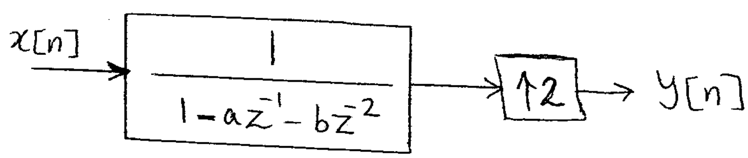
For parts (b) and (c) assume that the cost of each multiply per output sample $y[n]$ is 10, the cost of each compressor or expander block is 5, and the cost of each delay is 2.

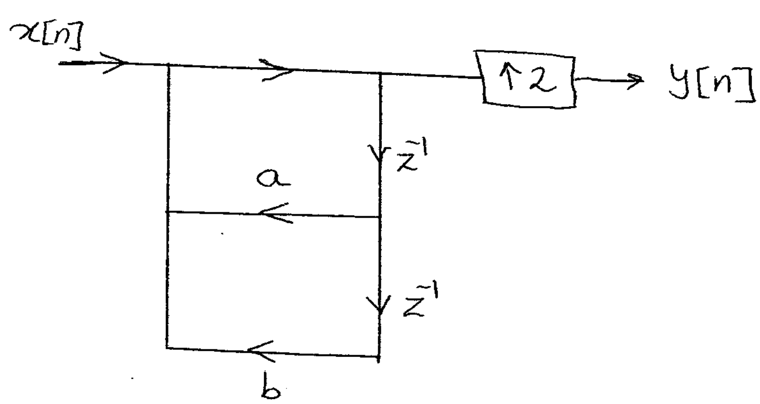
(c) (10%) If $H(z) = \dfrac{1}{1 - az^{-4} - bz^{-8}}$, determine the flowgraph or block diagram of the overall system which minimizes the implementation cost of the overall system in Figure 2-1, and the total cost of the implementation.





$\frac{2}{2} = 1$ multiply/output sample

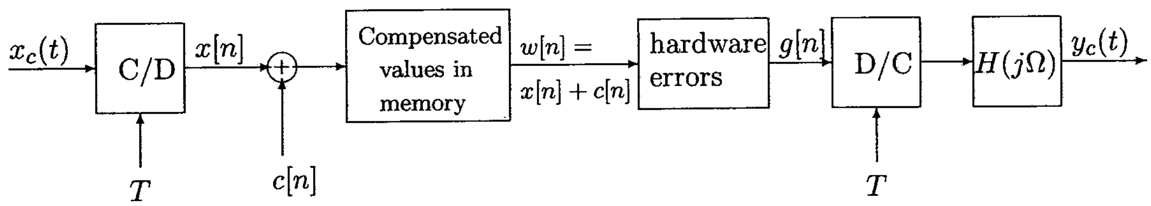




Total Cost: $1 \times 10 + 2 \times 2 + 1 \times 5 = 19$

Problem 3 (20%)

A bandlimited signal $x_c(t)$ has been sampled and the samples have been stored in memory. The samples are to be converted to a continuous-time signal $y_c(t)$ through an ideal C/D converter. It is known that because of a hardware fault, the input to the D/C will be faulty. The objective in this problem is to consider the possibility of pre-compensating $x[n]$ in memory so that the reconstructed signal $y_c(t)$ at the output will equal $x_c(t)$



$$
\begin{aligned}
X_c(j\Omega) &= 0, \text{ for } |\Omega| > \frac{2\pi}{3T} \\
w[n] &= x[n] + c[n] \\
X(e^{j\omega}) &= 0, \text{ for } \frac{2\pi}{3} < |\omega| < \pi \\
H(j\Omega) &= \begin{cases} 1 & \text{for } |\Omega| \le \frac{2\pi}{3T} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

(a) (10%) For this part assume that $c[n] = 0$ (i.e. there's no compensation and, consequently, $w[n] = x[n]$.) Assume further that the hardware errors have the effect of adding an undesired signal to $w[n]$, i.e. $g[n] = x[n] + e[n]$. Under what constraints on $e[n]$ or its Fourier transform $E(e^{j\omega})$ will $y_c(t) = x_c(t)$? Clearly explain your answer.

If $E(e^{j\omega}) = 0$ for $|\omega| < \frac{2\pi}{3}$ the LPF $H(j\Omega)$ will eliminate the error from the signal.

(b) (10%) For this part assume that the effect of the hardware error is to always force $g[0]$ to be zero. In designing the compensating signal we note that if $c[0] = -x[0]$ then $w[0] = 0$. Then the hardware error will not affect $w[n]$, i.e. $g[n] = w[n]$. With this in mind and considering your answer in (a), determine an appropriate choice for the compensating signal $c[n]$ so that $y_c(t) = x_c(t)$.

Any $c[n]$ that meets the following two criteria will work (solution is not unique)

1- $c[0] = -x[0]$

2- $c(e^{j\omega}) = 0 \quad |\omega| < \frac{2\pi}{3}$

One Example is : $c[n] = -x[0](-1)^n$

For more extensive discussion see:

http://www.rle.mit.edu/dspg/Pub_Conference.html

S. Dey, A. I. Russell, A. V. Oppenheim "Digital Pre-Compensation for faulty D/A Converters: The Missing Pixel Problem" In Procedings of the IEEE ICASSP, (Montreal), May 2004

Problem 4 (24%)

For this problem you may find useful the lecture slides which we have reproduced on page 14.

Consider the signal

$$s[n] = \alpha \left(\frac{2}{3}\right)^n u[n] + \beta \left(\frac{1}{4}\right)^n u[n]$$

where α and β are constants.

We wish to linearly predict $s[n]$ from its past p values using the relationship

$$\hat{s}[n] = \sum_{k=1}^{p} a_k s[n-k]$$

where the coefficients a_k are constants.

The coefficients a_k are chosen to minimize the prediction error

$$\mathcal{E} = \sum_{n=-\infty}^{+\infty} (s[n] - \hat{s}[n])^2$$

(a) (4%) With $\phi_s[m]$ denoting the autocorrelation of $s[n]$, write the equations for the case $p = 2$ the solution to which will result in a_1, a_2 (you do not need to derive the equations).

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] \\ \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \end{bmatrix}$$

(b) (10%) Determine a pair of values for α and β such that when $p = 2$, the solution to the normal equations is $a_1 = \frac{11}{12}$ and $a_2 = -\frac{1}{6}$. Is your answer unique? Explain.

$$S(z) = \alpha \frac{1}{1 - \frac{2}{3}z^{-1}} + \beta \frac{1}{1 - \frac{1}{4}z^{-1}} = \frac{\alpha - \frac{\alpha}{4}z^{-1} + \beta - \frac{2\beta}{3}z^{-1}}{(1 - \frac{2}{3}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

$$S(z) = \frac{\alpha + \beta - \frac{\alpha}{4}z^{-1} - \frac{2\beta}{3}z^{-1}}{1 - \frac{11}{12}z^{-1} + \frac{1}{6}z^{-2}}$$

If $a_1 = \frac{11}{12}$ and $a_2 = -\frac{1}{6}$, then we are modeling a 2^{nd} order all-pole signal. Therefore, the z^{-1} terms in the numerator must cancel.

$-\frac{\alpha}{4} = \frac{2\beta}{3}$, $\quad -3\alpha = 8\beta \longrightarrow$ One possibility : $\beta = -3$ & $\alpha = 8$

Solution is not unique, any pair $c\alpha$ and $c\beta$ with $c \neq 0$ will work.

(c) (10%) If $\alpha = 8$ and $\beta = -3$, determine the reflection coefficient k_3, resulting from using the Levinson recursion to solve the normal equations for $p = 3$. Is that different from k_3 when solving for $p = 4$? (An answer with no explanation will receive no credit.)

Since $s[n]$ is a 2^{nd} order all-pole signal, if you were to solve the Levinson recursion for $P=3$, then $K_3 = a_3^{(3)} = 0$. The K's do not change as the model order increases, therefore $K_3 = 0$ for any P.

END OF EXAM

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Levinson-Durbin Recursion

$$\sum_{k=1}^{p} a_k \phi_s[i-k] = \phi_s[i] \qquad i = 1,2,\ldots,p$$

$$T_p = \begin{bmatrix} \phi_s[0] & \phi_s[1] \cdots & \phi_s[p-1] \\ \phi_s[1] & \phi_s[0] \cdots & \\ \vdots & & \\ \phi_s[p-1] & \cdots & \phi_s[0] \end{bmatrix}$$

$$\alpha_p = [a_1, a_2 \ldots a_p]^T \qquad r_p = [\phi_s[1]\,\phi_s[2] \ldots \phi_s[p]]^T$$

$$\beta_p = [a_p, a_{p-1} \ldots a_1]^T \qquad \rho_p = [\phi_s[p]\,\phi_s[p-1] \ldots \phi_s[1]]^T$$

$$k_{p+1} = \frac{\phi_s[p+1] - (\rho_p)^T \alpha_p}{\phi_s[0] - (r_p)^T \alpha_p}$$

$$\varepsilon_p = -k_{p+1}\beta_p$$

$$\boxed{\alpha_{p+1} = \begin{bmatrix} \alpha_p \\ \text{-----} \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_p \\ \text{-----} \\ k_{p+1} \end{bmatrix} = \begin{bmatrix} a_1^{(p)} \\ a_2^{(p)} \\ \vdots \\ a_p^{(p)} \\ 0 \end{bmatrix} - k_{p+1} \begin{bmatrix} a_p^{(p)} \\ a_{p-1}^{(p)} \\ \vdots \\ a_1^{(p)} \\ -1 \end{bmatrix}} \qquad (7)$$

$a_k^{(p)}$ is the $a_k th$ coefficient for the pth order filter.