6.341: Discrete-Time Signal Processing

Fall 2005

Solutions for Problem Set 5

Issued: Thursday, October 13 2005.

Problem 5.1

(a),(b)

$$
\begin{aligned}
\epsilon^2 &= \frac{1}{2\pi} \int_{\pi}^{\pi} |H_d(e^{j\omega}) - H(e^{j\omega})|^2 d\omega \\
&= \sum_{n=-\infty}^{n=\infty} (h_d[n] - h[n])^2 \\
&= \sum_{n=-\infty}^{n=\infty} (h_d[n])^2 - \sum_{n=0}^{n=M} (h_d[n])^2 + \sum_{n=0}^{n=M} (h_d[n] - h[n])^2 \\
&\geq \underbrace{\sum_{n=-\infty}^{n=\infty} (h_d[n])^2 - \sum_{n=0}^{n=M} (h_d[n])^2}_{\text{constant w.r.t } h[n]}
\end{aligned}
$$

Thus,

$$ h[n] = h_d[n] \qquad 0 \leq n \leq M $$

must hold to minimize ϵ^2.

(c)

$$
w[n] = \begin{cases} 0 & n < 0 \text{ or } n > M \\ 1 & 0 \leq n \leq M \end{cases}
$$

Problem 5.2

(a)

$$
\begin{aligned}
\frac{Y_c(s)}{X_c(s)} &= \frac{A}{s+c} \\
\Rightarrow sY_c(s) + cY_c(s) &= AX_c(s) \\
\Rightarrow \frac{dy_c(t)}{dt} + cy_c(t) &= Ax_c(t)
\end{aligned}
$$

(b)

$$\frac{dy_c(t)}{dt}\bigg|_{t=nT} = Ax_c(nT) - cy_c(nT)$$

$$\frac{y_c(nT) - y_c((n-1)T)}{T} \approx Ax_c(nT) - cy_c(nT)$$

(c)

$$\frac{y_c(nT) - y_c((n-1)T)}{T} \approx Ax_c(nT) - cy_c(nT)$$

$$\frac{y[n] - y[n-1]}{T} = Ax[n] - cy[n]$$

$$y[n] - y[n-1] = TAx[n] - cTy[n]$$

$$y[n] + cTy[n] = TAx[n] + y[n-1]$$

$$(1 + cT - z^{-1})Y(z) = TAX(z)$$

$$H(z) = \frac{TA}{1 + cT - z^{-1}}$$

$$= \frac{A}{\frac{1-z^{-1}}{T} + c}$$

(d)

$$H_c(s)\big|_{s=\frac{1-z^{-1}}{T}} = \frac{A}{s+c}\bigg|_{s=\frac{1-z^{-1}}{T}}$$

$$= \frac{A}{\frac{1-z^{-1}}{T} + c}$$

$$= H(z)$$

(e)

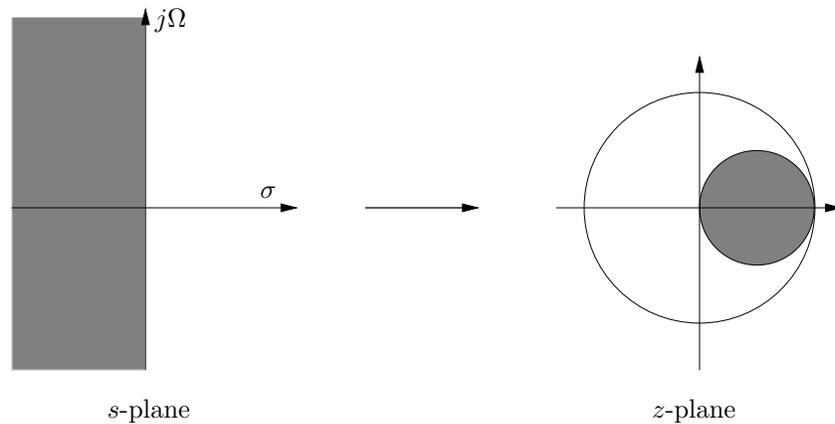$$s = \frac{1 - z^{-1}}{T}$$

$$z = \frac{1}{1 - sT}$$

For $s = \sigma + j\Omega$:

$$z = \frac{1}{1 - \sigma T - j\Omega T}$$

For $\sigma = 0$, $z = 1/(1 - j\Omega T)$. As Ω ranges from $-\infty$ to 0 to $+\infty$, z ranges from $j0^-$ to 1 to $j0^+$. For $\sigma \leq 0, |z| < 1$, thus if the continuous time system is stable and has poles in the left half plane, the discrete time approximation will be stable. The exact mapping is shown in the figure below.

s-plane $\qquad\qquad\qquad$ z-plane

The approximation is good only near the area where $z = 1$, i.e. $\Omega T \approx 0$. As T gets smaller the approximation holds for larger Ω.

(f) In this case $s = \frac{z-1}{T} \Rightarrow z = 1 + sT$. Thus, for $s = \sigma + j\Omega, z = 1 + \sigma T + j\Omega T$. In this case the left-half plane maps to the left of the line $\mathcal{R}e(z) = 1$. Therefore, a stable continuous time system might produce an unstable system with this approximation. Decreasing T might make the system stable (since a larger area from the s-plane is compressed inside the unit circle in the z-plane) but this is not guaranteed. As before, decreasing T improves the approximation.

s-plane $\qquad\qquad\qquad$ z-plane
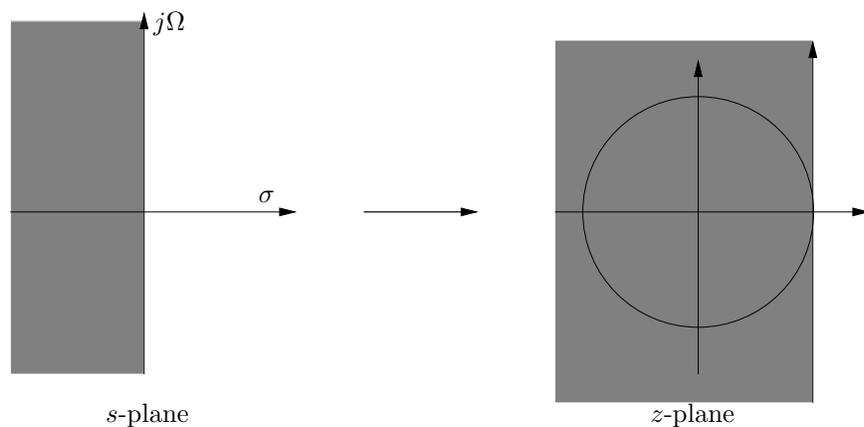
Problem 5.3

We know that the frequency response of the form

$$A_e(e^{j\omega}) = \sum_{k=0}^{L} a_k(\cos(\omega))^k$$

can have at most $L - 1$ local maxima and minima in the open interval $0 < \omega < \pi$ since it is in the form of a polynomial of degree L.

If we include all the endpoints of the region

$$\{0 \leq |\omega| \leq \omega_p\} \cup \{\omega_s \leq |\omega| \leq \pi\}$$

then we see we can have at most $L + 3$ alternation frequencies.

If the approximation does not decrease monotonically in transition band, and the transition band has two of the local maxima and minima of $A_e(e^{j\omega})$, then only $L - 3$ can be in the approximation bands. Even with all four endpoints of the approximation region as alternation points, we can only have a maximum of $L + 1$ alternation points. This does not satisfy the optimality condition of the Alternation Theorem which requires at least $L + 2$ alternation points. It follows that the transition band cannot have more than two local minima or maxima of $A_e(e^{j\omega})$ either.

If the transition only has one local maximum or minimum, the optimal approximation must also have a local maximum or minimum at ω_s or ω_p, since the optimal filter response must have alternations at ω_s and ω_p. If we add the four band edges to the remaining $L - 3$ maxima and minima in the approximation bands, we get $L + 1$ which is again too low.

Therefore, the transition band cannot have any local minima or maxima and must be monotonic.

Problem 5.4

For this filter $N = 3$ so the polynomial order L is $L = \frac{N-1}{2} = 1$.

Note that $h[n]$ must be a type-I FIR generalized linear phase filter, since it consists of three samples, and $H(e^{j\omega}) \neq 0$ for $\omega = 0$. $h[n]$ can therefore be written in the form

$$h[n] = a\delta[n] + b\delta[n-1] + a\delta[n-2]$$

Taking the DTFT of both sides gives

$$
\begin{aligned}
H(e^{j\omega}) &= a + be^{-j\omega} + ae^{-j2\omega} \\
&= e^{-j\omega}(ae^{j\omega} + b + ae^{-j\omega}) \\
&= e^{-j\omega}(b + 2a\cos(\omega)) \\
A(e^{j\omega}) &= b + 2a\cos(\omega)
\end{aligned}
$$

The filter must have at least $L+2 = 3$ alternations, but no more than $L+3 = 4$ alternations to satisfy the alternation theorem, and therefore be optimal in the minimax sense. Alternations

must occur at ω_p and ω_s. Three alternations can be obtained if ω_p , ω_s and π are alternation frequencies such that $A(e^{j\omega})$ undershoots at $\omega = \frac{\pi}{3}$, overshoots at $\omega = \frac{\pi}{2}$, and undershoots at $\omega = \pi$.

Let the error in the passband and stopband be δ. Then,

$$
\begin{aligned}
A(e^{j\omega})|_{\omega=\frac{\pi}{3}} &= 1-\delta &= b+a \\
A(e^{j\omega})|_{\omega=\frac{\pi}{2}} &= \delta &= b \\
A(e^{j\omega})|_{\omega=\pi} &= -\delta &= b-2a
\end{aligned}
$$

Solving this system of equations for a and b gives

$$
a = b = \frac{1}{3}
$$

Thus, the optimal (in the minimax sense) causal 3-point lowpass filter with the desired passband and stopband edge frequencies is

$$
h[n] = \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]
$$

Problem 5.5

(a)

$$
\begin{aligned}
h_1[n] &= h[n] + \delta_2\delta[n-n_0] \\
H_1(e^{j\omega}) &= H(e^{j\omega}) + \delta_2 e^{-jn_0\omega} \\
&= A_e(e^{j\omega})e^{-jn_0\omega} + \delta_2 e^{-jn_0\omega} \\
&= H_3(e^{j\omega})e^{-jn_0\omega} \\
\Rightarrow H_3(e^{j\omega}) &= A_e(e^{j\omega}) + \delta_2
\end{aligned}
$$

$A_e(e^{j\omega})$ is real and greater than $-\delta_2$. Thus $H_3(e^{j\omega})$ is real, nonnegative, and has zero phase.

(b) $H_3(e^{j\omega})$ has zero phase, and $h_3[n]$ (its inverse fourier transform) is real-valued. Thus, a zero at z_k implies there must also be zeros at $z_k^*, 1/z_k, 1/z_k^*$. In addition, a zero on the unit circle must be a double zero, since both the value of the frequency response and its derivative are zero. Thus, we factor the zeros inside the unit circle to $H_2(z)$ and the ones outside the unit circle to $H_2(1/z)$. The double zeros on the unit circle should be factored one to each $H_2(z)$ and $H_2(1/z)$. Since $H_2(z)$ only has zeros inside the unit circle, it is minimum phase (an exception is made here to allow zeros on the unit circle in the definition of minimum phase systems). Also, since $H_2(z)$ has zeros in conjugate pairs, $h_2[n]$ is real.

(c)

$$\begin{aligned} |H_{min}(e^{j\omega})|^2 &= \frac{H_2(e^{j\omega})H_2^*(e^{j\omega})}{a^2} \\ &= \frac{A_e(e^{j\omega}) + \delta_2}{a^2} \\ \Rightarrow |H_{min}(e^{j\omega})| &= \frac{\sqrt{A_e(e^{j\omega}) + \delta_2}}{a} \end{aligned}$$

A_e oscillates about 1 by $\pm\delta_1$ in the passband. Therefore,

$$\begin{aligned} \frac{\sqrt{1 - \delta_1 + \delta_2}}{a} &\leq |H_{min}(e^{j\omega})| \leq \frac{\sqrt{1 + \delta_1 + \delta_2}}{a} \\ -\delta_1' &\leq |H_{min}(e^{j\omega})| - 1 \leq +\delta_1' \\ \Rightarrow \delta_1' &= \frac{\sqrt{1 + \delta_1 + \delta_2} - \sqrt{1 - \delta_1 + \delta_2}}{2a} \end{aligned}$$

Similarly, A_e oscillates about 0 by $\pm\delta_2$ in the stopband. Therefore,

$$\begin{aligned} |H_{min}(e^{j\omega})| &\leq \frac{\sqrt{2\delta_2}}{a} \\ \Rightarrow \delta_2' &= \frac{\sqrt{2\delta_2}}{a} \end{aligned}$$

The spectral factorization reduces the order of the filter by half to $M/2$. Therefore, $h_{min}[n]$ has length $M/2 + 1$.

(d) The linear phase constraint ensures that the for every zero z, $1/z$ is also a zero. Thus we can factor $H_3(z)$ to get $H_2(z)$. Otherwise spectral factorization is not possible. Similarly, for Type II filters, n_0 is not an integer, so this technique is not possible.

Problem 5.6

(a) For no aliasing, we need $M\omega_s \leq \pi$. Therefore the maximum allowable M for no aliasing is $M = \pi/\omega_s$.

(b) $V(e^{j\omega})$: passband edge at $\omega = 0.9\pi/100$, stopband edge at $\omega = \pi/100$.

$Y(e^{j\omega})$: passband edge at $\omega = 0.9\pi$, stopband edge at $\omega = \pi$.
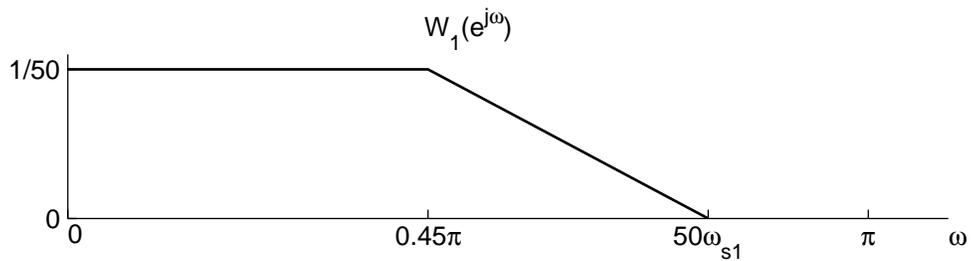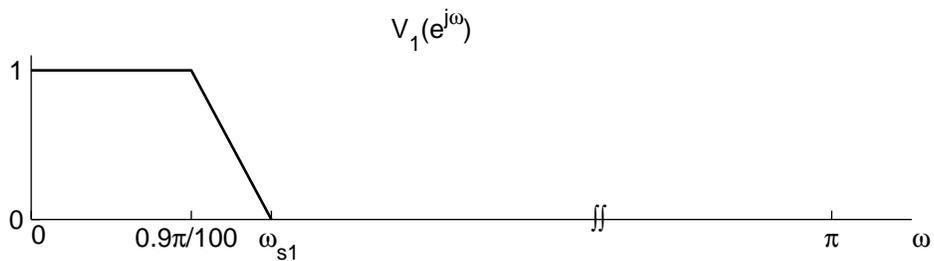
$V(e^{j\omega})$

$Y(e^{j\omega})$
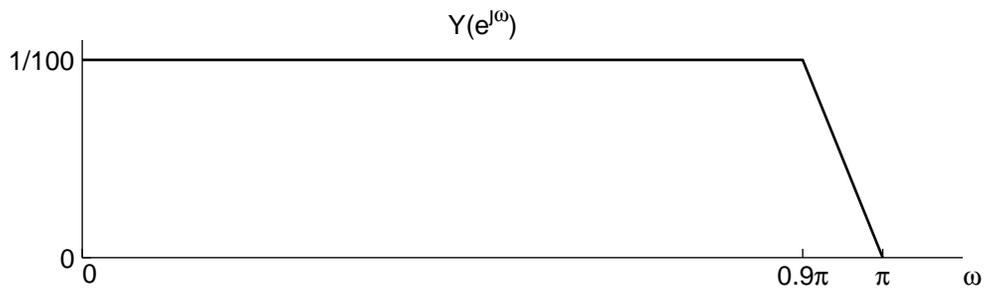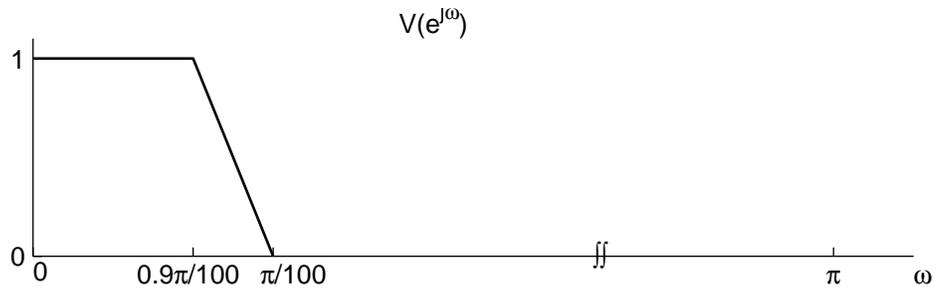
(c) $V_1(e^{j\omega})$: passband edge at $\omega = 0.9\pi/100$, stopband edge at $\omega = \omega_s$.

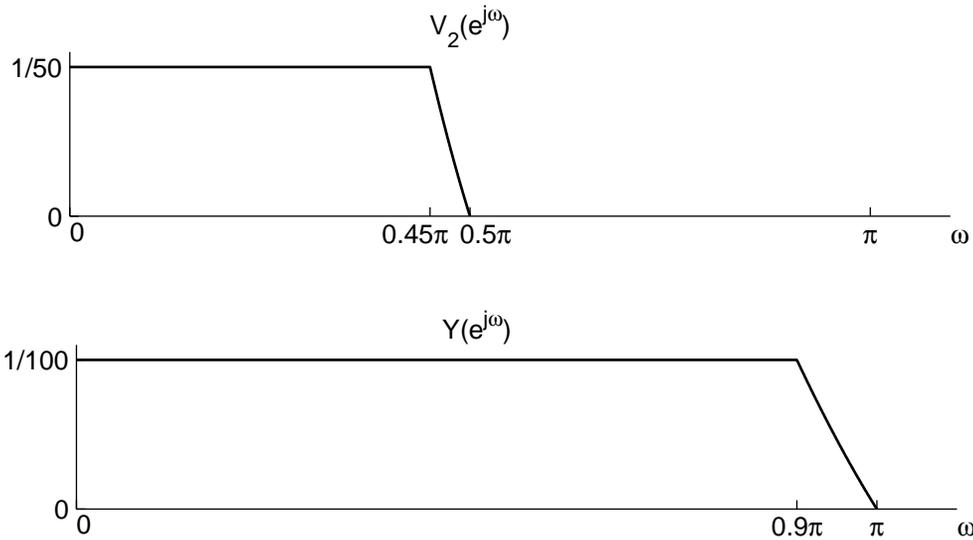$W_1(e^{j\omega})$: passband edge at $\omega = 0.9\pi/2$, stopband edge at $\omega = 50\omega_s$.

$V_2(e^{j\omega})$: passband edge at $\omega = 0.9\pi/2$, stopband edge at $\omega = \min(50\omega_{s1},\ \pi/2)$.

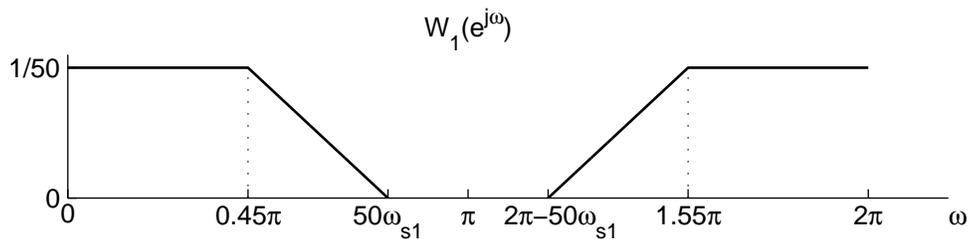$Y(e^{j\omega})$: passband edge at $\omega = 0.9\pi$, stopband edge at $\omega = \pi$.

$V_1(e^{j\omega})$

$W_1(e^{j\omega})$

For the case where $50\omega_{s1} = \frac{\pi}{2}$:

$V_2(e^{j\omega})$

$Y(e^{j\omega})$

(d) Since filter $H_w(e^{j\omega})$ is 2π-periodic, we need to consider the stopband edge from the replication around $\omega = 2\pi$. This edge can extend all the way to $\omega = \omega_{p1}$ but no further:

$$50\omega_{p1} \leq 2\pi - 50\omega_{s1}$$

The maximum value is $\omega_{s1} = 3.1\frac{\pi}{100}$.

$W_1(e^{j\omega})$

(e) $N \approx 5068.7$. Use $N = 5069$. In a direct implementation this would mean $5069 \cdot 100$ multiplications to compute each sample of $y[n]$. However, the direct implementation is very inefficient, and we can take advantage of the fact that multipliers and downsamplers commute, as illustrated in the figure below:

This way, the number of multiplications per sample of $y[n]$ reduces to 5069. Also, taking advantage of the symmetry of the impulse response, we can use a symmetric structure as discussed in Section 6.5.3 in OSB, and the number of multiplications to compute each sample of $y[n]$ further goes down to 2535. Note that we did not use the polyphase
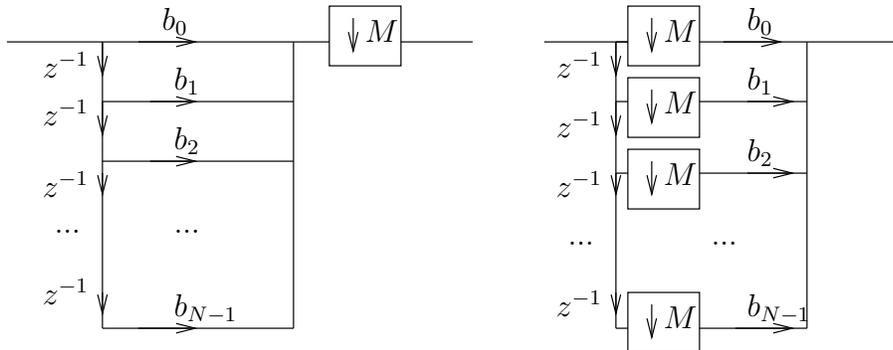
Figure 5.6-1: Reducing the number of multiplications.

implementation of the low pass filter, we have just changed the order of multiplies and the downsamplers. In the polyphase representation, the delays would be grouped in such a way that they can be interchanged with the downsamplers as well.

(f) Using the value of ω_{s1} from part (d), the transition bandwidth is $2.2\pi/100$. For $H_1(e^{j\omega})$ we need $N_1 \approx 231.4$. Note that if we had not extended the stopband edge ω_{s1} as far as possible but used $\pi/100$ instead, then this filter alone would have required $N_1 \approx 5068.7$, the same as part (e). Instead we can use $N_1 = 232$. Using impulse response symmetry, we need 116 multiplications for each sample of $v_1[n]$. In a direct implementation this would translate into $116 \cdot 50$ multiplications for each sample of $w_1[n]$ and $116 \cdot 100$ multiplications to compute each sample of $y[n]$ (contribution of H_1 alone). However, changing the order of multiplies and downsampling as in part (e), we will have 116 multiplications for each sample of $w_1[n]$ and $116 \cdot 2$ for each sample of $y[n]$ (the second downsampler still remains on the path to $y[n]$).

$N_2 \approx 102.3$, use $N_2 = 103$. Using impulse response symmetry, we need 52 multiplications for each sample of $v_2[n]$. In a direct form implementation that would translate into $2 \cdot 52$ multiplications for each sample of $y[n]$ (contribution of H_2 alone). Changing the order of multiplies and downsamplers, we need only 52 multiplications for each sample of $y[n]$.

Putting these together, the total is $2 \cdot 116 + 52 = 284$ multiplications to compute each sample of $y[n]$ in the steady state.

(g) $N_1 \approx 250.1$. Use $N_1 = 251$. Using impulse response symmetry, and switching the order of downsamplers and multiplies, we need 126 multiplications for each sample of $v_1[n]$ and also for each sample of $w_1[n]$.

$N_2 \approx 110.6$. Use $N_2 = 111$. Using impulse response symmetry, and switching the order of downsamplers and multipliers, we need 56 multiplications for each sample of $v_2[n]$ and also for each sample of $y[n]$.

The total is $2 \cdot 126 + 56 = 308$ multiplications to compute each sample of $y[n]$.

(h) No, we do not need to change the specification in the stop band.

Problem 5.7

(a) $A_e(e^{j\omega})$ has 7 alternations of the error. The approximation bands are of equal length and the weighting function is unity in both bands, yet the stopband has one more alternation than the passband. If it were an optimal filter, it would not. We can negate $A_e(e^{j\omega})$, add 1 in the frequency domain, and shift it by π (by multiplying the impulse response by $(-1)^n$) to obtain a different lowpass filter that meets the same specifications. Since the optimal approximation is unique, the one shown in the figure cannot be optimal.

(b) A polynomial of degree L can have at most $L - 1$ local minima or maxima in an open interval. Since $A_e(e^{j\omega})$ has 3 local extrema in the interval from $0 < \omega < \pi$, we know $L \geq 4$.

Problem 5.8

(a)

$$
\begin{aligned}
H_{\text{eff}}(j\Omega) &= \frac{1}{T}H(e^{j\Omega T})H_0(j\Omega)H_r(j\Omega) \\
&= \frac{\sin(\Omega T/2)}{\Omega T/2}H(e^{j\Omega T})e^{-j\Omega T/2}, |\Omega| < \pi/T
\end{aligned}
$$

(b) There is a 51/2=25.5 samples fractional delay due to the linear phase system and a $T/2$ delay due to $h_0(t)$ Therefore the total delay is 2.6ms.

(c) $H(e^{j\omega}) = e^{-j\omega M/2}\cos(\omega/2)P(\cos\omega)$, where $M = 51$ and $P(\cos\omega) = \sum_{k=0}^{L} a_k(\cos\omega)^k$.

The phase term cannot be compensated for, but the desired frequency response should compensate for the effects of $H_0(j\Omega)$ in the passband. We also account for the presence of the factor $\cos(\omega/2)$ in $H(e^{j\omega})$, so the function to be approximated by the polynomial $P(\cos\omega)$ is :

$$
H_d(e^{j\omega}) = \begin{cases} \frac{\omega/2}{\sin(\omega/2)\cos(\omega/2)}, & |\omega| \leq 0.2\pi \\ 0, & 0.4\pi \leq \omega \leq \pi \end{cases}
$$

The overall response should be equiripple, but any ripple is multiplied by $H_0(j\Omega)$ and the $\cos(\omega/2)$ term. Thus the weight should be scaled appropriately:

$$
W(\omega) = \begin{cases} \frac{\sin(\omega/2)\cos(\omega/2)}{\omega/2}, & |\omega| \leq 0.2\pi \\ \frac{\sin(\omega/2)\cos(\omega/2)}{\omega/2}, & 0.4\pi \leq \omega \leq \pi \end{cases}
$$

(d) Both $H_d(e^{j\omega})$ and $W(\omega)$ should be adjusted in the passband to compensate for the slope. Note that we cannot compensate for the phase of $H_r(j\Omega)$:

$$H_d(e^{j\omega}) = \begin{cases} \frac{\omega/2}{\sin(\omega/2)\cos(\omega/2)|H_r(j\omega/T)|}, & |\omega| \leq 0.2\pi \\ 0, & 0.4\pi \leq \omega \leq \pi \end{cases}$$

$$W(\omega) = \begin{cases} \frac{\sin(\omega/2)\cos(\omega/2)|H_r(j\omega/T)|}{\omega/2}, & |\omega| \leq 0.2\pi \\ \frac{\sin(\omega/2)\cos(\omega/2)|H_r(j\omega/T)|}{\omega/2}, & 0.4\pi \leq \omega \leq \pi \end{cases}$$