6.262: Discrete Stochastic Processes 4/11/11

Lecture 17: Countable-state Markov chains

Outline:

- Strong law proofs
- Positive-recurrence and null-recurrence
- Steady-state for positive-recurrent chains
- Birth-death Markov chains
- Reversibility

Let $\{Y_i;\, i \geq 1\}$ be the IID service times for a (G/G/∞) queue and let $\{N(t); t > 0\}$ be the renewal process with interarrivals $\{X_i;\, i \geq 1\}$. Consider the following plausability argument for $\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{N(t,\omega} Y_i(\omega)$.

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{N(t,\omega)} Y_i(\omega) = \lim_{t\to\infty} \left[\frac{N(t,\omega)}{t} \, \frac{\sum_{i=1}^{N(t,\omega)} Y_i(\omega)}{N(t,\omega)} \right] \quad (1)$$

$$= \lim_{t\to\infty} \frac{N(t,\omega)}{t} \lim_{t\to\infty} \frac{\sum_{i=1}^{N(t,\omega)} Y_i(\omega)}{N(t,\omega)} \quad (2)$$

$$= \lim_{t\to\infty} \frac{N(t,\omega)}{t} \lim_{n\to\infty} \frac{\sum_{i=1}^{n} Y_i(\omega)}{n} \quad (3)$$

$$= \frac{1}{\overline{\overline{X}}} \overline{Y} \qquad \textbf{WP1} \quad (4)$$

This assumes $\overline{X} < \infty$, $\overline{Y} < \infty$.

To do this carefully, work from bottom up.

Let $A_1 = \{\omega : \lim_{t\to\infty} N(t,\omega)/t = 1/\overline{X}\}$. By the strong law for renewal processes $\Pr\{A_1\} = 1$.

Let $A_2 = \{\omega : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i(\omega) = \overline{Y}\}$. By the SLLN, $\Pr\{A_2\} = 1$. Thus (3) = (4) for $\omega \in A_1A_2$ and $\Pr\{A_1A_2\} = 1$.

Assume $\omega \in A_2$, and $\epsilon > 0$. Then $\exists m(\epsilon,\omega)$ such that $|\frac{1}{n}\sum_{i=1}^{n} Y_i(\omega) - \overline{Y}| < \epsilon$ for all $n \geq m(\epsilon,\omega)$. If $\omega \in A_1$ also, then $\lim_{t\to\infty} N(t,\omega) = \infty$, so $\exists t(\epsilon,\omega)$ such that $N(t,\omega) \geq m(\epsilon,\omega)$ for all $t \geq t(\epsilon,\omega)$.

$$\left|\frac{\sum_{i=1}^{N(t,\omega)} Y_i(\omega)}{N(t,\omega)} - \overline{Y}\right| < \epsilon \qquad \text{for all } t \geq t(\epsilon,\omega)$$

Since ϵ is arbitrary, (2) = (3) = (4) for $\omega \in A_1A_2$.

Finally, can we interchange the limit of a product of two functions (say $f(t)g(t)$) with the product of the limits? If the two functions each have finite limits (as the functions of interest do for $\omega \in A_1A_2$), the answer is yes, establishing (1) = (4).

To see this, assume $\lim_t f(t) = a$ and $\lim_t g(t) = b$. Then

$$\begin{aligned} f(t)g(t) - ab &= (f(t)-a)(g(t)-b) + a(g(t)-b) + b(f(t)-a) \\ |f(t)g(t) - ab| &\leq |f(t)-a||g(t)-b| + |a||g(t)-b| + |b||f(t)-a| \end{aligned}$$

For any $\epsilon > 0$, choose $t(\epsilon)$ such that $|f(t) - a| \leq \epsilon$ for $t \geq t(\epsilon)$ and $|g(t) - b| \leq \epsilon$ for $t \geq t(\epsilon)$. Then

$$|f(t)g(t) - ab| \leq \epsilon^2 + \epsilon|a| + \epsilon|b| \qquad \text{for } t \geq t(\epsilon).$$

Thus $\lim_t f(t)g(t) = \lim_t f(t) \lim_t g(t)$.

Review - Countable-state chains

Two states are in the same class if they communicate (same as for finite-state chains).

Thm: All states in the same class are recurrent or all are transient.

Pf: Assume j is recurrent; then $\sum_n P_{jj}^n = \infty$. For any i such that $j \leftrightarrow i$, $P_{ij}^m > 0$ for some m and P_{ji}^ℓ for some ℓ. Then (recalling $\lim_t \mathsf{E}[N_{ii}(t)] = \sum_n P_{ii}^n$)

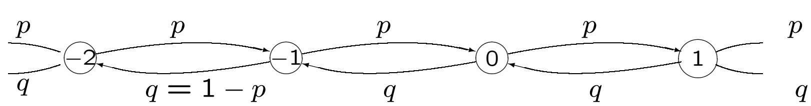
$$\sum_{n=1}^{\infty} P_{ii}^n \geq \sum_{k=n-m-\ell}^{\infty} P_{ij}^m P_{jj}^k P_{jk}^\ell = \infty$$

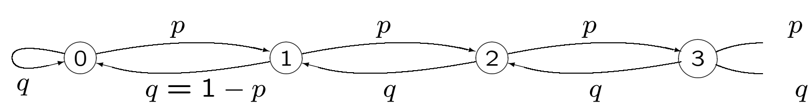
By the same kind of argument, if $i \leftrightarrow j$ are recurrent, then $\sum_{n=1}^{\infty} P_{ij}^n = \infty$ (so also $\lim_t \mathsf{E}\left[N_{ij}^t\right] = \infty$).

If a state j is recurrent, then the recurrence time T_{jj} might or might not have a finite expectation.

Def: If $\mathsf{E}\left[T_{jj}\right] < \infty$, j is positive-recurrent. If T_{jj} is a rv and $\mathsf{E}\left[T_{jj}\right] = \infty$, then j is null-recurrent. Otherwise j is transient.

For $p = 1/2$, each state in each of the following is null recurrent.





Positive-recurrence and null-recurrence

Suppose $i \leftrightarrow j$ are recurrent. Consider the renewal process of returns to j with $X_0 = j$. Consider rewards $R(t) = 1$ whenever $X(t) = i$. By the renewal-reward thm (4.4.1),

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t R(\tau)d\tau = \frac{\mathsf{E}[\mathsf{R}_n]}{\overline{T}_{jj}} \qquad \textbf{WP1},$$

where $\mathsf{E}[R_n]$ is the expected number of visits to i within a recurrence of j. The left side is $\lim_{t\to\infty} \frac{1}{t} N_{ji}(t)$, which is $1/\overline{T}_{ii}$. Thus

$$\frac{1}{\overline{T}_{ii}} = \frac{\mathsf{E}[R_n]}{\overline{T}_{jj}}$$

Since there must be a path from j to i, $\mathsf{E}[R_n] > 0$.

Thm: For $i \leftrightarrow j$ recurrent, either both are positive-recurrent or both null-recurrent.

Steady-state for positive-recurrent chains

We define steady-state probabilities for countable-state Markov chains in the same way as for finite-state chains, namely,

Def: $\{\pi_i; i \geq 0\}$ is a steady-state distribution if

$$\pi_j \geq 0; \ \pi_j = \sum_i \pi_i P_{ij} \quad \textbf{for all } j \geq 0 \quad \textbf{and } \sum_j \pi_j = 1$$

Def: An irreducible Markov chain is a Markov chain in which all pairs of states communicate.

For finite-state chains, irreducible means recurrent. Here it can be positive-recurrent, null-recurrent, or transient.

If steady-state $\boldsymbol{\pi}$ exists and if $\Pr\{X_0 = i\} = \pi_i$ **for each** i**, then** $\mathsf{p}_{X_1}(j) = \sum_i \pi_i P_{ij} = \pi_j$**. Iterating,** $\mathsf{p}_{X_n}(j) = \pi_j$**, so steady-state is preserved. Let** $\widetilde{N}_j(t)$ **be number of visits to** j **in** $(0, t]$ **starting in steady state. Then**

$$\mathsf{E}\left[\widetilde{N}_j(t)\right] = \sum_{k=1}^{n} \Pr\{X_k = j\} = n\pi_j$$

Awkward thing about renewals and Markov: $\widetilde{N}_j(t)$ **works for some things and** $N_{jj}(t)$ **works for others. Here is a useful hack:**

$N_{ij}(t)$ **is 1 for first visit to** j **(if any) plus** $N_{ij}(t) - 1$ **for subsequent recurrences** j **to** j**. Thus**

$$\begin{aligned} \mathsf{E}\left[N_{ij}(t)\right] &\leq 1 + \mathsf{E}\left[N_{jj}(t)\right] \\ \mathsf{E}\left[\widetilde{N}_j(t)\right] &= \sum_i \pi_i \mathsf{E}\left[N_{ij}(t)\right] \leq 1 + \mathsf{E}\left[N_{jj}(t)\right] \end{aligned}$$

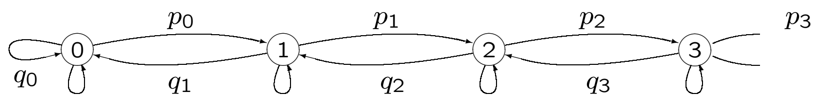
Major theorem: For an irreducible Markov chain, the steady-state equations have a solution if and only if the states are positive-recurrent. If a solution exists, then $\pi_i = 1/\overline{T}_{ii} > 0$ **for all** i**.**

Pf: (only if; assume $\boldsymbol{\pi}$ **exists, show positive-recur.) For each** j **and** t**,**

$$\begin{aligned} \pi_j &= \frac{\mathsf{E}\left[\widetilde{N}_j(t)\right]}{t} \leq \frac{1}{t} + \frac{\mathsf{E}\left[N_{jj}(t)\right]}{t} \\ &\leq \lim_{t\to\infty} \frac{\mathsf{E}\left[N_{jj}(t)\right]}{t} = \frac{1}{\overline{T}_{jj}} \end{aligned}$$

Since $\sum_j \pi_j = 1$**, some** $\pi_j > 0$**. Thus** $\lim_{t\to\infty} \mathsf{E}\left[N_{jj}(t)\right]/t > 0$ **for that** j**, so** j **is positive-recurrent. Thus all states are positive-recurrent. See text to show that '$\leq$' above is equality.**

Birth-death Markov chains



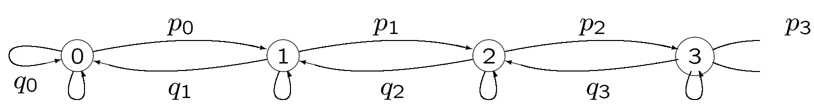
For any state i and any sample path, the number of $i \to i+1$ transitions is within 1 of the number of $i+1 \to j$ transitions; in the limit as the length of the sample path $\to \infty$,

$$\pi_i p_i = \pi_{i+1} q_{i+1}; \qquad \pi_{i+1} = \frac{\pi_i p_i}{q_{i+1}}$$

Letting $\rho_i = p_i / q_{i+1}$, this becomes

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \rho_j; \qquad \pi_0 = \frac{1}{1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \rho_j}.$$

This agrees with the steady-state equations.



$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \rho_j; \qquad \pi_0 = \frac{1}{1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \rho_j}.$$

This solution is a function only of $\rho_0, \rho_1, \ldots$ and doesn't depend on size of self loops.

The expression for π_0 converges (making the chain positive recurrent) (essentially) if the ρ_i are asymptotically less than 1.

Methodology: We could check renewal results carefully to see if finding π_i by up/down counting is justified. Using the major theorem is easier.

Birth-death chains are particularly useful in queuing where births are arrivals and deaths departures.

Reversibility

$$\Pr\Big\{X_{n+k}, ...X_{n+1}|X_n, ...X_0\Big\} = \Pr\Big\{X_{n+k}, ..., X_{n+1} \mid X_n\Big\}$$

For any A^+ defined on X_{n+1} up and A^- defined on X_{n-1} down,

$$\Pr\Big\{A^+ \mid X_n, A^-\Big\} = \Pr\Big\{A^+ \mid X_n\Big\}$$

$$\Pr\Big\{A^+, A^- \mid X_n\Big\} = \Pr\Big\{A^+ \mid X_n\Big\}\Pr\Big\{A^- \mid X_n\Big\}.$$

$$\Pr\Big\{A^- \mid X_n, A^+\Big\} = \Pr\Big\{A^- \mid X_n\Big\}.$$

$$\Pr\Big\{X_{n-1} \mid X_n, X_{n+1}, \dots, X_{n+k}\Big\} = \Pr\{X_{n-1} \mid X_n\}.$$

By Bayes,

$$\Pr\{X_{n-1} \mid X_n\} = \frac{\Pr\{X_n \mid X_{n-1}\}\Pr\{X_{n-1}\}}{\Pr\{X_n\}}.$$

If the forward chain is in steady state, then

$$\Pr\{X_{n-1} = j \mid X_n = i\} = P_{ji}\pi_j/\pi_i.$$

Aside from the homogeniety involved in starting at time 0, this says that a Markov chain run backwards is still Markov. If we think of the chain as starting in steady state at time $-\infty$, these are the equations of a (homogeneous) Markov chain. Denoting $\Pr\{X_{n-1} = j \mid X_n = i\}$ as the backward transition probabilities P^*_{ji}, forward/ backward are related by

$$\pi_i P^*_{ij} = \pi_j P_{ji}.$$

Def: A chain is reversible if $P^*_{ij} = P_{ij}$ for all i, j.

Thm: A birth/death Markov chain is reversible if it has a steady-state distribution.