

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.262 Discrete Stochastic Processes

Midterm Exam - Solutions

April 7, 2009

Problem 1

1a) (i) Recall that two states i and j in a Markov chain communicate if each is accessible from the other, *i.e.*, if there is a walk from each to the other. Since all transitions move from left to right, each state is accessible only from those to the left, and therefore no state communicates with any other state. Thus each state is in a class by itself.

States 0 to 5 (and thus the classes $\{0\}, \dots, \{5\}$) are each transient since each is inaccessible from an accessible state (*i.e.*, there is a path away from each from which there is no return). State 6 is recurrent. States 1 and 6 (and thus class $\{1\}$ and $\{6\}$) are each aperiodic since $P_{00}^1 \neq 0$ and $P_{66}^1 \neq 0$. The periods of classes $\{1\}$ to $\{5\}$ are undefined, but no points were taken off if some other answer was given for these periods.

1a) (ii) Each state on the circle on the left communicates with all other states on the left and similarly for the circle on the right. Since there is a transition from left to right, and also from right to left, the entire set of states communicate, so there is single class containing all states. Note that there is a cycle of length 6 starting in state 0 and passing through state 1, and there is a cycle of length 8 from 0 going around the left circle. The greatest common divisor of 6 and 8 is 2. Thus, the chain is either periodic with period 2 or aperiodic. The easiest way to see that the period is 2, *i.e.*, all cycles have even length, is to renumber the states, going from 0 to 7 in order on the left, and 8 to 15 on the right, with even states going only to odd and odd going only to even. Thus the chain must be periodic with period 2.

1a) (iii) The entire set of states communicate by virtually the same argument as in 1a ii. State 0 has a cycle of length 2 through state 1 and of length 7 via the left circle. The greatest common divisor of 2 and 7 is 1, so state 1 has period 1. The chain is then aperiodic since all states in a class have the same period.

1b) It may be helpful to review Exercise 3.25 before reading this solution. We first show that the intervals (in number of coin-flips) between successive occurrences of HHTHTHTT are IID. To see this, let Y_1, Y_2, \dots be the sequence of heads and tails and let X_1, X_2, \dots be the intervals between successive occurrences of the string HHTHTHTT (viewing the time of occurrence of a string as the time of the final coin flip in the string).

For any given sample values $X_1 = x_1, \dots, X_{i-1} = x_{i-1}$, we see that X_i is the interval from $x_1 + \dots + x_{i-1}$ until the next occurrence of HHTHTHTT. This interval must be at least 8 since the string does not overlap with any delayed replica of itself (*i.e.*, no prefix of the string is the same as the corresponding suffix of the same length). This interval depends only on the coin-toss sequence $\{Y_n; n > x_1 + \dots + x_{i-1}\}$ and this sequence has the same probabilistic description as the original sequence for all choices of x_1, \dots, x_{i-1} . Thus X_i is independent of X_1, \dots, X_{i-1} , and this is true for all $i > 1$, making $X_i; i \geq 1$ a renewal process.

As in Exercise 3.25, this result depends on the no prefix property of HHTHTHTT, *i.e.*, the property that no prefix is the same as the suffix of the same length. For strings without the prefix property, $X_i; i \geq 1$ is a delayed renewal process.

For this renewal process, a renewal occurs at toss n , for $n \geq 8$, if the string HHTHTHTT occurs from $n - 7$ to n . This event has probability 2^{-8} . Letting $m(t) = \mathbf{E}[N(t)]$ be the expected number of renewals up to time t , we see that $m(t) = (t - 7)2^{-8}$. By the elementary renewal theorem, then, the expected length of an interrenewal period is $\mathbf{E}[X] = 2^8$.

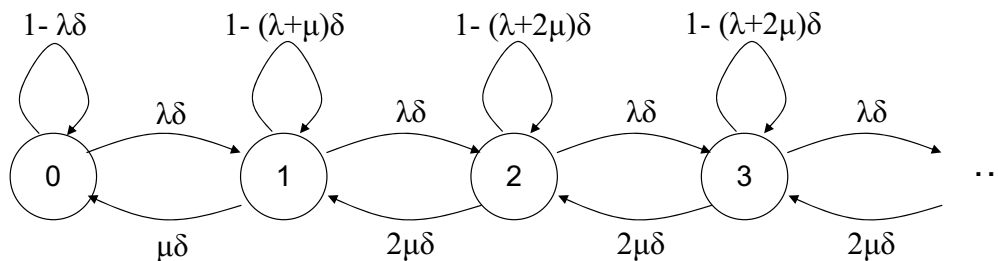
1c) The simple way to do this is to view $N_1(t)$ and $N_2(t)$ as a splitting of a Poisson process of rate $s + f$. Then each arrival to the combined process is directed to process 1 with probability $f/(f + s)$ and to process 2 with probability $s/(f + s)$. These Bernoulli choices are independent of the interarrival times. Thus, interarrival interval Z_n is independent of which split process that arrival goes to, and Z_n has density $f_{Z_n}(z) = (f + s) \exp(-(f + s)t)$.

This result might seem counterintuitive, since one might think that an arrival that is split to the slow process takes longer to arrive. If one visualizes a sequence of combined arrivals, with most going to the fast process and an occasional one to the slow process, and remembers that the combined process is Poisson, then the result here seems more intuitive.

Question 2 (36 pts)

The following solution concerns the reading of the problem suggested during the exam, namely that as soon as Alice becomes free, Bob transfers his customer to her, and goes back to drinking coffee. However, regardless of the interpretation, the system (in terms of the arrival and departure processes) corresponds to an $M/M/2$ queue.

- a Let each state $\{0, 1, 2, \dots\}$ denote the total number of customers in the store, either talking to Bob or Alice or waiting in line. Choosing δ sufficiently small, the probability of having two or more customer arrivals (or two or more departures or an arrival and a departure, etc.) in a period of length δ vanishes. The probability of observing a customer arrival becomes $\lambda\delta$, that of observing a customer departure when both Bob and Alice are working becomes $2\mu\delta$ and that of observing a customer departure when Alice alone is working becomes $\lambda\delta$. The sampled-time Markov chain description of the system is the following.



Note that this is a birth-death chain.

- b All states communicate, therefore the chain is composed of a single class. There is at least one self-loop, which implies that the corresponding state has period 1, which in turn implies that the rest of the class (and therefore chain) is aperiodic.

There are several ways to show that the chain is positive recurrent.

Solution 1 We compute the steady-state probabilities π_0, π_1, \dots . It then suffices to show that $\pi_i > 0$ for each $i = 0, 1, \dots$, or, alternatively, show that $\pi_i > 0$ for some $i = 0, 1, \dots$ and note that if one state is positive recurrent, the containing class will be as well.

The steady-state equations are given by:

$$\begin{aligned}\pi_0 &= (1 - \lambda\delta)\pi_0 + \mu\delta\pi_1 \\ \pi_1 &= \lambda\delta\pi_0 + (1 - (\lambda + \mu)\delta)\pi_1 + 2\mu\delta\pi_2 \\ \pi_2 &= \lambda\delta\pi_1 + (1 - (\lambda + 2\mu)\delta)\pi_2 + 2\mu\delta\pi_3 \\ &\vdots \\ 1 &= \pi_0 + \pi_1 + \pi_2 + \dots\end{aligned}$$

Alternatively, a reduced set of equations is obtained noting that in steady state, the fraction of incoming transitions to a state equals that of outgoing transitions, and thus:

$$\begin{aligned}\pi_0\lambda\delta &= \pi_1\mu\delta \\ \pi_i\lambda\delta &= 2\pi_{i+1}\mu\delta, \quad \text{for } i \geq 1 \\ 1 &= \pi_0 + \pi_1 + \pi_2 + \dots\end{aligned}$$

(Note that the above steady-state equations for a birth-death chain are derived as Eqn. 5.27 in the notes.)

The two systems of equations are equivalent and either will yield:

$$\pi_i = \left(\frac{1}{2}\right)^{i-1} \left(\frac{\lambda}{\mu}\right)^i \pi_0, \quad i \geq 1$$

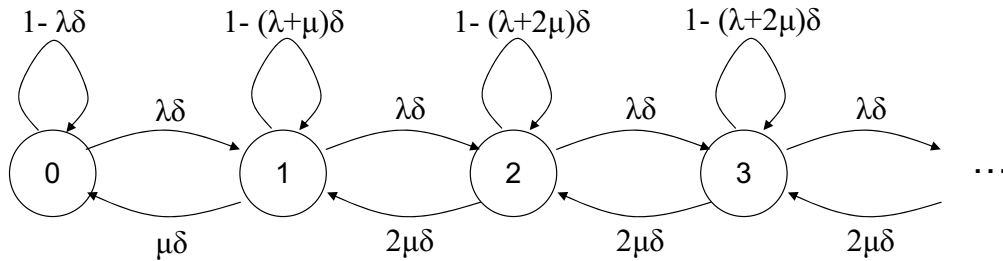
and therefore

$$\pi_0 = \frac{1}{1 + \sum_{i=1}^{\infty} (1/2)^{i-1} \left(\frac{\lambda}{\mu}\right)^i} = \frac{2\mu - \lambda}{2\mu + \lambda} > 0.$$

Solution 2 Though the steady-state probabilities will become useful later in the problem, a way to show positive recurrence bypassing the above calculation is the following. Let $X_n = 1$ and suppose that the process leaves state 1 at time $k > n$. It suffices to show that the process eventually returns to state 1 with probability 1. It will follow that state 1 is positive recurrent, from which it will follow that all the states in the containing class, and therefore the entire chain, are positive recurrent.

First note that starting from state 1, the next state must be either 0 or 2. Because the chain is birth/death, the chain cannot get from 0 to states larger than 1 without returning to 1, and cannot get from states larger than 1 to 0 without returning to 1. Thus, for questions of recurrence, one can analyze whether the chain is recurrent by separately asking whether the chain with only states 0 and 1 is recurrent and whether the chain with states 1 and greater is recurrent.

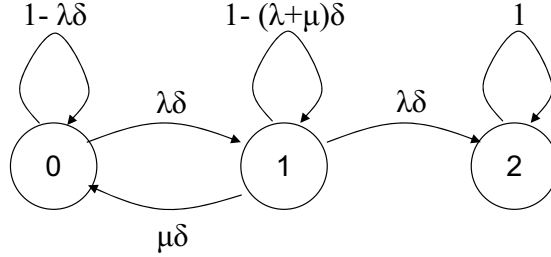
Assuming that the process leaves state 1 at time $k > n$ and conditioned on $X_k = 0$, the process returns to state 1 eventually w.p.1, as truncating the chain at state 1 yields a finite Markov chain with a single recurrent class. Now condition on $X_k = 2$ and consider the following chain:



Note that the modification does not affect the probability of eventually returning to state 1 after entering state 2. However, the above is a birth-death chain has a transition probability $p = \lambda\delta$ of moving forward and transition probability $q = 2\mu\delta$ of moving backward for all transitions. Since $2\mu > \lambda$, the chain is positive recurrent and the process returns to state 1 eventually w.p.1. It follows that state 1 is positive recurrent.

- c Starting from time 0, let T_B denote the time until Bob first becomes busy. There are again several possible ways to approach this problem.

Solution 1: In the sampled-time Markov chain representation, the period of time until Bob first becomes busy corresponds to the number of transitions to reach state 2 from state 0, additionally taking into account the fact that every transition lasts δ units of time. It therefore suffices to compute the expected first passage time $\bar{T}_{0,2}$, from which it will follow that $E(T_B) = \delta\bar{T}_{0,2}$. One way to do this is to compute the time to absorption, starting at state 0, for the following chain:



The corresponding equations become

$$\begin{aligned}\bar{T}_{0,2} &= (1 - \lambda\delta)(\bar{T}_{0,2} + 1) + (\lambda\delta)(\bar{T}_{1,2} + 1) = 1 + (1 - \lambda\delta)\bar{T}_{0,2} + (\lambda\delta)\bar{T}_{1,2} \\ \bar{T}_{1,2} &= (1 - (\lambda + \mu)\delta)(\bar{T}_{1,2} + 1) + \mu\delta(\bar{T}_{0,2} + 1) + \lambda\delta(1 + \bar{T}_{2,2}) = 1 + (1 - (\lambda + \mu)\delta)\bar{T}_{1,2} + \mu\delta\bar{T}_{0,2}, \\ \bar{T}_{2,2} &= 0\end{aligned}$$

which yields

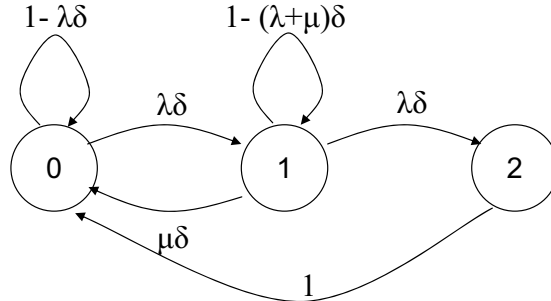
$$\bar{T}_{1,2} = \bar{T}_{0,2} - \frac{1}{\lambda\delta} = \frac{1 + \mu\delta}{(\lambda + \mu)\delta}\bar{T}_{0,2}$$

and thus,

$$\boxed{\bar{T}_{0,2} = \frac{\mu}{\lambda^2\delta} + \frac{2}{\lambda\delta}}.$$

Since the expected number of transitions to reach state 2 from state 0 is $\bar{T}_{0,2}$, and each transition corresponds to δ minutes, it follows that the expected time for Bob to start helping is $(\bar{T}_{0,2})\delta = \frac{\mu}{\lambda^2} + \frac{2}{\lambda} = \frac{20}{9} + \frac{20}{3} = \frac{80}{9}$ minutes. Note that the answer does not depend on δ .

Solution 2: Compute the expected length of time between two returns to state 2 for the following chain:



The steady-state equations become:

$$\begin{aligned}\pi_0 &= (1 - \lambda\delta)\pi_0 + \mu\delta\pi_1 + \pi_2 \\ \pi_1 &= \lambda\delta\pi_0 + (1 - (\lambda + \mu)\delta)\pi_1 \\ \pi_2 &= \lambda\delta\pi_1 \\ 1 &= \pi_1 + \pi_2 + \pi_3.\end{aligned}$$

The solution yields

$$\pi_0 = (1 + \mu/\lambda)\pi_1, \quad \pi_1 = \frac{1}{2 + \lambda\delta + \mu/\lambda}, \quad \pi_2 = \lambda\delta\pi_1.$$

From renewal theory, we have that starting from state 2, the expected time to return to 2 is $1/\pi_2$. It follows that starting from 0, the expected number of transitions to first reach state 2 is given by $1/\pi_2 - 1$. Since each transition corresponds to δ minutes, we have

$$\delta\bar{T}_{0,2} = (1/\pi_2 - 1)\delta = \frac{\mu}{\lambda^2} + \frac{2}{\lambda},$$

which is what we had before.

Solution 3 (Due to SLH, MY, DB) Let $T_B = Y_1 + Y_2$, where Y_1 denotes the time of the first customer arrival after time zero and Y_2 is the remaining time (beginning with the arrival of the first customer) until Bob first becomes busy. Clearly, $E(Y_1) = 1/\lambda$. From here on, the next “event” can either be the departure from Alice’s customer or an arrival of yet another customer. Since both times are given by independent exponential random variables, we can view the setup as two competing exponentials or (equivalently) as an arrival to a merged process. Letting $A = \{\text{customer 1 departs before customer 2 arrives}\}$ and $C = \{\text{customer 2 arrives before customer 1 departs}\}$, we have that $P(A) = \mu/(\lambda + \mu)$ and $P(C) = \lambda/(\lambda + \mu)$. By the Total Expectation Theorem,

$$E(T_B) = \frac{1}{\lambda} + \frac{\mu}{\lambda + \mu}E(Y_2 | A) + \frac{\lambda}{\lambda + \mu}E(Y_2 | C).$$

Now, $E(Y_2 | C) = 1/(\lambda + \mu)$ since Bob becomes busy the moment the second customer (which is an arrival to the merged process) arrives. In contrast, under A , the system restarts probabilistically the moment the first customer departs since the arrival time of the second customer is given by a memoryless r.v. Thus, $E(Y_2 | C) = 1/(\lambda + \mu) + E(T_B)$. Putting things together and solving for $E(T_B)$, we again obtain that

$$E(T_B) = \frac{\mu}{\lambda^2} + \frac{2}{\lambda}.$$

- d The moment Bob stops helping Alice corresponds to the first time the chain is in state 1 after being in state 2. From that point on, the time until Bob is again needed to help Alice is given by the time to reach state 2 from state 1. The desired answer is

$$\boxed{\delta\bar{T}_{1,2} = \frac{\mu}{\lambda^2} + \frac{1}{\lambda} = \frac{50}{9} \text{ min.}},$$

computed in any of the following ways.

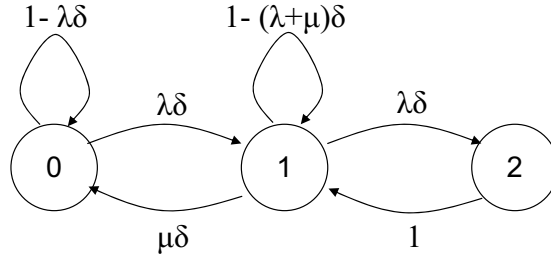
Solution 1 $\bar{T}_{1,2}$ is obtained in the process of solving for $\bar{T}_{1,2}$ in Solution 1 to part c).

Solution 2 $\bar{T}_{1,2}$ is obtained from $\bar{T}_{0,2}$ obtained by either Solution 1 or Solution 2 to part c), by noting that the expected time to transition from state 0 to state 1 is $1/(\lambda\delta)$ (expectation of a geometric random variable with success probability $\lambda\delta$). Thus, $\bar{T}_{0,2} = \bar{T}_{1,2} + 1/(\lambda\delta)$ and therefore $\delta\bar{T}_{1,2} = \delta\bar{T}_{0,2} - 1/\lambda$.

Solution 3 Analogously to the Solution 3 to part c), conditioning yields

$$\bar{T}_{1,2} = \frac{\lambda}{\lambda + \mu} \frac{1}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} \frac{1}{\lambda + \mu} + \bar{T}_{0,2} \quad .$$

Solution 4 $\bar{T}_{1,2} = 1/\pi_2 - 1$ in the following modified chain (note that the process renews every time state 2 is reached):



The steady-state equations become:

$$\begin{aligned} \pi_0 &= (1 - \lambda\delta)\pi_0 + \mu\delta\pi_1 \\ \pi_1 &= \lambda\delta\pi_0 + (1 - (\lambda + \mu)\delta)\pi_1 + \pi_2 \\ \pi_2 &= \lambda\delta\pi_1 \\ 1 &= \pi_1 + \pi_2 + \pi_3. \end{aligned}$$

The solution yields

$$\pi_0 = \frac{1}{1 + \lambda/\mu + \lambda^2\delta/\mu}, \quad \pi_1 = \frac{\lambda}{\mu}\pi_0, \quad \pi_2 = \frac{\lambda^2}{\mu}\delta\pi_0$$

and thus

$$\bar{T}_{1,2} = 1/\pi_2 - 1 = \frac{\mu}{\lambda^2\delta} + \frac{1}{\lambda\delta}.$$

- e The fraction of time Bob spends helping Alice is given by the fraction of time the chain of part a) spends in states $\{2, 3, \dots\}$. Renewal theory tells us that the corresponding long-term fraction of time equals $\pi_2 + \pi_3 + \dots = 1 - \pi_0 - \pi_1$ (Strong Law for renewal rewards +

Blackwell's Theorem), where the steady-state probabilities again refer to the chain of part a). Substituting the steady-state probabilities calculated in b),

$$1 - \pi_0 - \pi_1 = 1 - \pi_0(1 + \lambda/\mu) = \frac{\lambda^2}{\mu(2\mu + \lambda)} = \frac{9}{14}.$$

f Let us first consider which type of answer would make intuitive sense. In a birth/death chain, with renewals on a given transition, the expected time between renewals should be the reciprocal of the probability of that renewal. The period between renewals is then one segment in the upper part of the chain and one segment in the lower part, where the fraction of time in each is equal to the long term fraction of time spent in each region. Making this argument rigorous is the essence of **Solution 1**, but, as always, several different approaches are possible.

Solution 1 From part e), the fraction of time that Bob is busy is given by

$$F_b = \frac{3\lambda\mu - \lambda^2}{\mu(2\mu + \lambda)} = \frac{9}{14}.$$

Defining a renewal every time Bob becomes free and letting $R(t) = 1$ during times where Bob is helping Alice and $R(t) = 0$ otherwise, the Strong Law for Renewal Rewards yields

$$F_b = \frac{E(R_1)}{E(X_1)},$$

where X_1 is the length of the first renewal period. The probability of having a renewal at time n , for large n , is given by $2\mu\delta\pi_2$. By Blackwell's Theorem (as the process has span δ), it follows that

$$E(X_1) = \frac{1}{2\mu\pi_2} = \frac{\mu}{\lambda^2} \frac{2\mu + \lambda}{2\mu - \lambda} = \frac{140}{9},$$

and thus

$$E(R_1) = \frac{9}{14} \frac{140}{9} = 10 \text{ min.}$$

Solution 2 As an alternative method of finding $E(X_1)$, notice that, from part d), once Bob becomes free he may expect to remain so for the next $\frac{\mu}{\lambda^2} + \frac{1}{\lambda} = \frac{50}{9}$ minutes on average. On the other hand, once he starts helping, he may expect to do so for the next $E(R_1)$ on average. Therefore,

$$F_b = \frac{3\lambda\mu - \lambda^2}{\mu(2\mu + \lambda)} = \frac{9}{14} = \frac{E(R_1)}{\frac{\mu}{\lambda^2} + \frac{1}{\lambda} + E(R_1)} = \frac{E(R_1)}{E(R_1) + \frac{50}{9}},$$

which yields

$$E(R_1) = 10 \text{ min.}$$

Solution 2' A variant on Solution 2 consists of keeping the same renewal process define two rewards: $R(t)$ taking value 1 over periods where Bob is busy and $\tilde{R}(t)$ taking value 1 over periods where Bob is free. Notice that

$$F_b = \lim_{t \rightarrow \infty} \frac{\int_0^t R(\tau) d\tau}{t} = 1 - \lim_{t \rightarrow \infty} \frac{\int_0^t \tilde{R}(\tau) d\tau}{t}.$$

Since $E(R_1) + E(\tilde{R}_1) = E(X_1)$ and since $E(\tilde{R}_1)$ was found to equal $50/9$ in part d), it then follows that

$$E(R_1) = \frac{F_b}{1 - F_b} E(\tilde{R}_1) = \frac{9}{5} \frac{50}{9} = 10 \text{ min.}$$

Note on Solutions 1 - 2:

Letting $R(t) = 1$ during periods Bob is busy, a number of students noticed that

$$\lim_{t \rightarrow \infty} \frac{\int_0^t R(\tau) d\tau}{t} = 1 - \pi_0 - \pi_1 = E(R_1)/E(X_1).$$

However, a large proportion of those students also defined the underlying renewal in a way that the expected accumulated reward over one renewal, $E(R_1)$, does not give us what we asked for. We need the expected length of any period Bob spends helping Alice. If we define a renewal every time the process hits state 0 (for instance), a potentially large number of renewals will incur a total reward of zero, thus reducing the value of $E(R_1)$. The only two viable options are to define a reward the moment Bob becomes busy (on the transition from 1 to 2), or, alternatively, the moment Bob becomes free (on the transition from 2 to 1). Can you see why both approaches yield answer?

Solution 3 (Due to HSK) Suppose at time n , the process has just entered state 2 from state 1. Let define a collection of random variables $\{Y_k\}_{k=1}^{\infty}$ as follows:

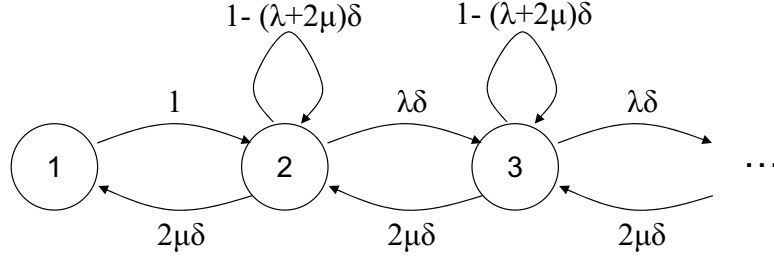
$$Y_k = \begin{cases} 0 & \text{if } X_{n+k} = X_{n+k-1} \\ 1 & \text{if } X_{n+k} = X_{n+k-1} + 1 \\ -1 & \text{if } X_{n+k} = X_{n+k-1} - 1 \end{cases}$$

Let $N = \min\{n \mid X_1 + \dots + X_n \leq 1\}$. Note that since the chain is positive recurrent, we have that N is both finite with probability 1 and also $E(N) < \infty$. Furthermore the event $\{N \geq n\} = \{X_1 > 1, \dots, X_{n-1} > 1\}$ is independent of X_n, X_{n+1} , etc. It follows that N is a valid stopping time. By Wald's equality, we then have that

$$-1 = E(N)E(Y_1) = E(N)(\lambda\delta(1) + 2\mu\delta(-1) + 0) \implies E(N) = \frac{1}{(2\mu - \lambda)\delta},$$

and the answer is given by $\delta E(N) = 10$ min. Awesome stuff!

Solution 4 We're looking for the expected amount of time that the chain of part a) spends in states $\{2, 3, \dots\}$ once it has entered state 2. Accordingly, defining a renewal every time a chain enters state 1, it suffices to look at the expected time in states $\{2, 3, \dots\}$ for the following chain:



The length of time in states $\{2, 3, \dots\}$ is given by $\delta(1/\pi_1 - 1)$, similarly to the previous argument of this type in part c). The steady-state equations for this chain are given by:

$$\begin{aligned}
 \pi_1 &= 2\mu\delta\pi_2 \\
 \pi_2 &= \pi_1 + (1 - (\lambda + 2\mu)\delta)\pi_2 + 2\mu\delta\pi_3 \\
 \pi_3 &= \lambda\delta\pi_2 + (1 - (\lambda + 2\mu)\delta)\pi_3 + 2\mu\delta\pi_4 \\
 &\vdots \\
 1 &= \pi_1 + \pi_2 + \pi_3 + \dots,
 \end{aligned}$$

or, alternatively:

$$\begin{aligned}
 \pi_1 &= 2\mu\delta\pi_2 \\
 \pi_i\lambda\delta &= \pi_{i+1}2\mu\delta \quad \text{for } i \geq 2 \\
 1 &= \pi_1 + \pi_2 + \pi_3 + \dots,
 \end{aligned}$$

Both sets of equations yield

$$\begin{aligned}
 \pi_2 &= \frac{1}{2\mu\delta}\pi_1 \\
 \pi_k &= \frac{\lambda}{2\mu}\pi_{k-1} = \frac{1}{2\mu\delta} \frac{\lambda}{2\mu}^{k-2} \pi_1, \quad k \geq 3,
 \end{aligned}$$

and therefore

$$\pi_1 = \frac{1}{1 + \sum_{k=2}^{\infty} \frac{1}{2\mu\delta} \left(\frac{\lambda}{2\mu}\right)^{k-2}} = \frac{1}{1 + \frac{1}{2\mu\delta} \frac{1}{1 - \frac{\lambda}{2\mu}}} = \frac{\delta(2\mu - \lambda)}{\delta(2\mu - \lambda) + 1}.$$

It follows that the expected amount of time that Bob is busy helping Alice is given by

$$\delta \left(\frac{1}{\pi_1} - 1 \right) = \frac{1}{2\mu - \lambda} = 10 \text{ min.}$$

III. (40 pts) A small production facility builds widgets. Widgets require two subassemblies, aidjets and bidjets. The time to build an aidjet is a nonnegative rv A with density $f_A(t)$ and distribution function $F_A(t)$. Successive aidjets require IID construction intervals. The time to build a bidjet is also a nonnegative rv B with density $f_B(t)$ and distribution function $F_B(t)$. Successive bidjet times are also IID. Also aidjet and bidjet times are independent of each other. In this question, you can either choose f_A and f_B to be uniform over $(0, 2]$ and calculate a numerical answer or leave them abstract and provide a formula.

a) Initially the facility is set up with an aidjet facility and a bidjet facility but no storage. Thus the first aidjet and the first bidjet both start construction at time 0, but the first to finish stops and waits until the other is finished. The widget is then produced in zero extra time and each facility starts on the next part. This continues ad infinitum. Let $N_1(t)$ be the number of widgets produced by time t .

a1) (5 pts) Is $N_1(t)$ a renewal counting process?

Solution: $N_1(t)$ is a renewal counting process since each interval for producing a widget is independent of the others, and the time to produce a widget is $\max(A, B)$, which is a rv since a and b are each rv's.

a2) (5 pts) Find the time-average number of widgets produced in the limit $t \rightarrow \infty$ and state carefully what that time-average means.

Solution: By the strong law for renewal processes, the limiting time-average, with probability 1, is $1/\mathbf{E}[W_1]$, where the rv $W_1 = \max(A_1, B_1)$ is the time to construct the first widget. Note that $\mathbf{P}\{W_1 \leq t\} = \mathbf{P}\{A_1 \leq t\} \mathbf{P}\{B_1 \leq t\}$. Thus

$$\begin{aligned} F_W(t) &= F_A(t)F_B(t) \\ \mathbf{E}[W_1] &= \int_0^\infty [1 - F_A(t)F_B(t)] dt \\ \lim_{t \rightarrow \infty} \frac{N_1(t)}{t} &= \left[\int_0^\infty [1 - F_A(t)F_B(t)] dt \right]^{-1} \quad \text{W.P.1} \end{aligned}$$

where we evaluated $\mathbf{E}[W_1]$ by integrating the complementary distribution function and then used the strong law for renewal processes.

For the uniform distribution, the above integral is $4/3$, so the time average number of widgets per unit time is 0.75 .

b) Now some storage is provided and two aidjets are produced one after the other and, starting at the same time, two bidjets are produced one after the other. Whichever finishes a pair first stops and waits for the other to finish a pair. The first widget is produced when both have finished one part and the second widget when both have finished the second part. When both finish their second part, both start again, and this continues ad infinitum. Let $N_2(t)$ be the number of widgets by time t in this new scheme.

b1) (6 pts) Is $N_2(t)$ a renewal counting process? If not, describe a renewal counting process that accomplishes the same purpose.

Solution: $N_2(t)$ is not a renewal counting process since the inter-renewal time to produce the second widget is dependent on how much of a head start one of the facilities has over the other in the production of the second widget, and this head start depends on the previous widget time. If we let $N_2^{(2)}$ be the number of pairs of widgets produced by time t , then this is a renewal counting process for the same reason that $N_1(t)$ is a renewal process in part (a)

b2) (6 pts) Show that $N_1(t) \leq N_2(t)$ assuming that the sample times $A_1(\omega), A_2(\omega), \dots$ for building successive aidjets are the same in scheme 1 and 2. Similarly the sample times for bidjets, $B_1(\omega), B_2(\omega), \dots$ are the same in each scheme. Hint: this is not an asymptotic result - look at the first pair of widgets.

Solution: First consider an example. Suppose, for a given sample point ω , that $A_1(\omega)$ is very small, say 0.1, and $A_2(\omega)$ is very large, say 1.9. Suppose also that $B_1(\omega) = B_2(\omega) = 1$. Then in scheme 2, the first two aidjets and the first 2 bidjets are completed at time 2, so the first two widgets are completed at time 2. In scheme 1, the aidjet facility stops at time 0.1 and waits for the first bidjet at time 1. This means that the second aidjet is completed at time 2.9, so the second widget is completed later in the first scheme than the second.

In general, first assume that $A_1(\omega) < B_1(\omega)$. Then the second aidjet in scheme 1 does not start construction until time $B_1(\omega)$. The second aidjet is completed at $B_1(\omega) + A_2(\omega)$. This is greater than $A_1(\omega) + A_2(\omega)$, which is the completion time of the second aidjet in scheme 2. The completion of the second bidjet occurs at the same time in schemes 1 and 2, so the second widget is completed either at the same time or earlier in scheme 2. The same argument applies when $A_1(\omega) \geq B_1(\omega)$.

Another approach is to say that waiting occurs in scheme 1 for either aidjets or bidjets between the first and second assembly, and no such waiting occurs for scheme 2. All other times are the same in the two schemes.

b3) (6 pts) Find the time-average number of widgets produced in the limit $t \rightarrow \infty$ and state carefully what that time-average means.

Solution: Let $f_{AA}(t)$ be $f_A(t) * f_A(t)$ and $f_{BB}(t) = f_B(t) * f_B(t)$. Let $F_{AA}(t)$ and $F_{BB}(t)$ be the corresponding distribution functions. Then the distribution function for the time to produce the first pair of widgets, say $F_{WW}(t)$ is

$$\begin{aligned} F_{WW}(t) &= F_{AA}(t)F_{BB}(t) \\ \mathbf{E}[WW] &= \int_0^{\infty} [1 - F_{AA}(t)F_{BB}(t)] dt \end{aligned}$$

where $\mathbf{E}[WW]$ is the expected time to construct a pair of widgets in scheme 2. After much tedious but elementary integration, this is 2.4667 in scheme 2, somewhat less than 2.667 for scheme 1.

Then $\lim_{t \rightarrow \infty} N_2^{(2)}(t)/t$, i.e., the time average number of widget pairs per unit time, is $1/\mathbf{E}[WW]$. It follows that the number of widgets per unit time is

$$\lim_{t \rightarrow \infty} \frac{N_2(t)}{t} = \frac{2}{\mathbf{E}[WW]} = 2 \int_0^{\infty} [1 - F_{AA}(t)F_{BB}(t)] dt^{-1}$$

This is 0.811 widgets per time unit for the uniform distribution.

c) Now assume that neither facility ever stops and waits; they continue producing aidgets and bidgets, which are paired as available and immediately are combined into widgets. Let $N_\infty(t)$ be the number of widgets produced by time t

c1) (6 pts) Explain carefully why $N_\infty(t)$ is not a renewal counting process.

Solution: At any given time t , the aidget and bidget processes are at various times in their production cycles, and the probability that both finish a unit simultaneously is zero (since they both have probability densities). The time until the next widget thus depends on how far into the production cycle each are. One could try to use a pair of ages, one for each facility as a renewal point, but because the construction intervals are given by densities, there is zero probability that any given pair of ages will be repeated.

c2) (6 pts) Find the time-average number of widgets produced in the limit $t \rightarrow \infty$ and state carefully what that time-average means.

Solution: Both subassemblies form renewal processes individually, and $N(t) = \min(N_A(t), N_B(t))$. Since $N_1(t)/t$ and $N_2(t)/t$ each have limits W.P.1, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{N_\infty(t)}{t} &= \lim_{t \rightarrow \infty} \min \frac{N_A(t)}{t} \frac{N_B(t)}{t} \\ &= \min \frac{\lim_{t \rightarrow \infty} N_a(t)}{t}, \frac{\lim_{t \rightarrow \infty} N_b(t)}{t} \\ &= \min \frac{1}{\mathbb{E}[A(t)]} \frac{1}{\mathbb{E}[B(t)]} = \frac{1}{\max(\mathbb{E}[A(t)], \mathbb{E}[B(t)])} \end{aligned}$$

For the uniform distribution, this is a widget per time unit.

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6.262 Discrete Stochastic Processes
Spring 2011

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