LECTURE SLIDES ON

CONVEX ANALYSIS AND OPTIMIZATION

BASED ON 6.253 CLASS LECTURES AT THE

MASS. INSTITUTE OF TECHNOLOGY

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Based on the book

"Convex Optimization Theory," Athena Scientific, 2009, including the on-line Chapter 6 and supplementary material at

http://www.athenasc.com/convexduality.html

LECTURE 1
AN INTRODUCTION TO THE COURSE

LECTURE OUTLINE

- The Role of Convexity in Optimization
- Duality Theory
- Algorithms and Duality
- Course Organization

HISTORY AND PREHISTORY

- Prehistory: Early 1900s - 1949.
  - Caratheodory, Minkowski, Steinitz, Farkas.
  - Properties of convex sets and functions.

- Fenchel - Rockafellar era: 1949 - mid 1980s.
  - Duality theory.
  - Minimax/game theory (von Neumann).
  - (Sub)differentiability, optimality conditions, sensitivity.

- Modern era - Paradigm shift: Mid 1980s - present.
  - Nonsmooth analysis (a theoretical/esoteric direction).
  - Algorithms (a practical/high impact direction).
  - A change in the assumptions underlying the field.

OPTIMIZATION PROBLEMS

- Generic form:

$$\begin{aligned} &\text{minimize} \;\; f(x) \\ &\text{subject to} \;\; x \in C \end{aligned}$$

Cost function $f : \Re^n \mapsto \Re$, constraint set C, e.g.,

$$\begin{aligned} C = X &\cap \big\{x \mid h_1(x) = 0, \ldots, h_m(x) = 0\big\} \\ &\cap \big\{x \mid g_1(x) \le 0, \ldots, g_r(x) \le 0\big\} \end{aligned}$$

- Continuous vs discrete problem distinction
- Convex programming problems are those for which f and C are convex
  - They are continuous problems
  - They are nice, and have beautiful and intuitive structure
- However, convexity permeates all of optimization, including discrete problems
- Principal vehicle for continuous-discrete connection is duality:
  - The dual problem of a discrete problem is continuous/convex
  - The dual problem provides important information for the solution of the discrete primal (e.g., lower bounds, etc)

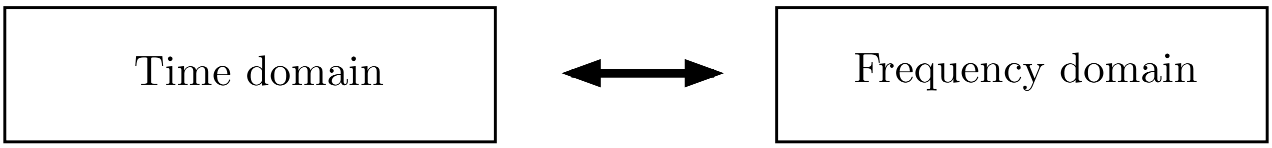
WHY IS CONVEXITY SO SPECIAL?

- A convex function has no local minima that are not global
- A nonconvex function can be “convexified” while maintaining the optimality of its global minima
- A convex set has a nonempty relative interior
- A convex set is connected and has feasible directions at any point
- The existence of a global minimum of a convex function over a convex set is conveniently characterized in terms of directions of recession
- A polyhedral convex set is characterized in terms of a finite set of extreme points and extreme directions
- A real-valued convex function is continuous and has nice differentiability properties
- Closed convex cones are self-dual with respect to polarity
- Convex, lower semicontinuous functions are self-dual with respect to conjugacy

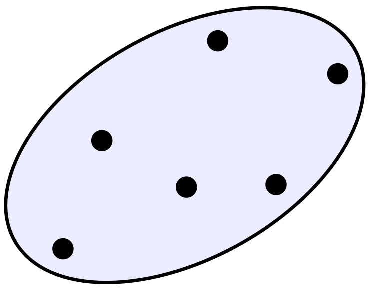
DUALITY

- Two different views of the same object.

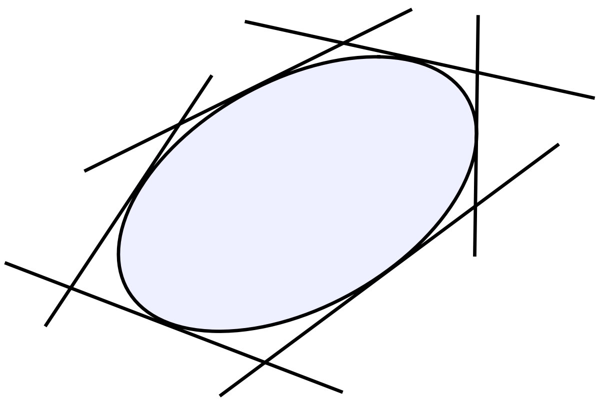
- Example: Dual description of signals.



- Dual description of **closed** convex sets



A union of points



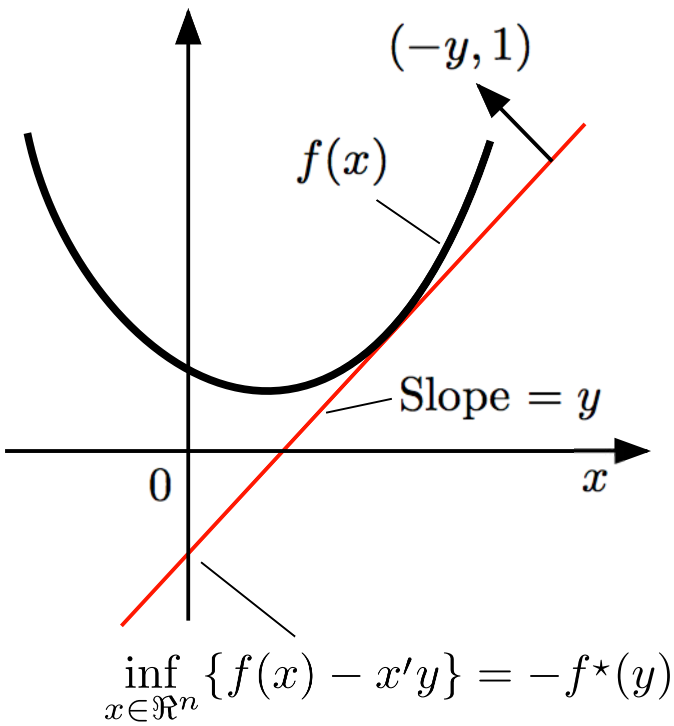
An intersection of halfspaces

DUAL DESCRIPTION OF CONVEX FUNCTIONS

- Define a closed convex function by its epigraph.

- Describe the epigraph by hyperplanes.

- Associate hyperplanes with crossing points (the conjugate function).



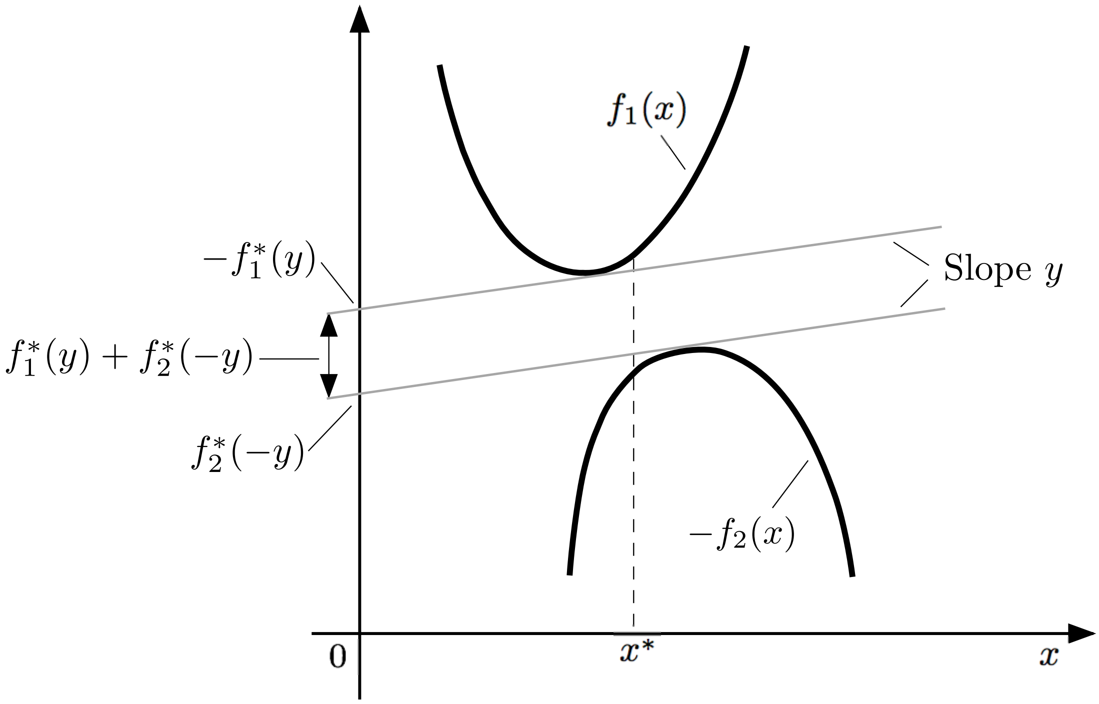
Primal Description

Values $f(x)$

Dual Description

Crossing points $f^*(y)$

FENCHEL PRIMAL AND DUAL PROBLEMS



Primal Problem Description
Vertical Distances

Dual Problem Description
Crossing Point Differentials

- Primal problem:

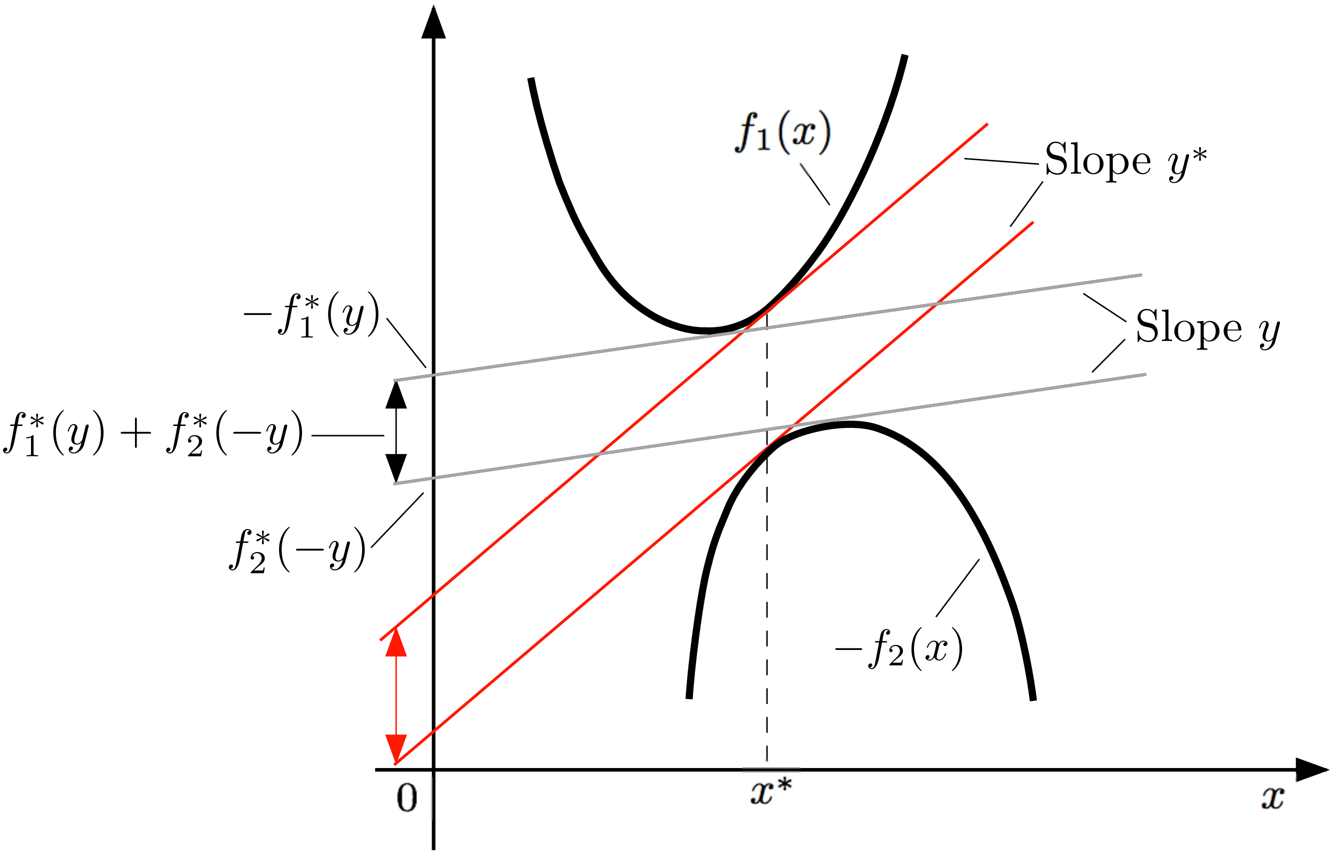
$$\min_x \big\{ f_1(x) + f_2(x) \big\}$$

- Dual problem:

$$\max_y \big\{ - f_1^*(y) - f_2^*(-y) \big\}$$

where f_1^* and f_2^* are the conjugates

FENCHEL DUALITY



$$\min_x \{ f_1(x) + f_2(x) \} = \max_y \{ -f_1^\star(y) - f_2^\star(-y) \}$$

- Under favorable conditions (convexity):
  - The optimal primal and dual values are equal
  - The optimal primal and dual solutions are related

A MORE ABSTRACT VIEW OF DUALITY

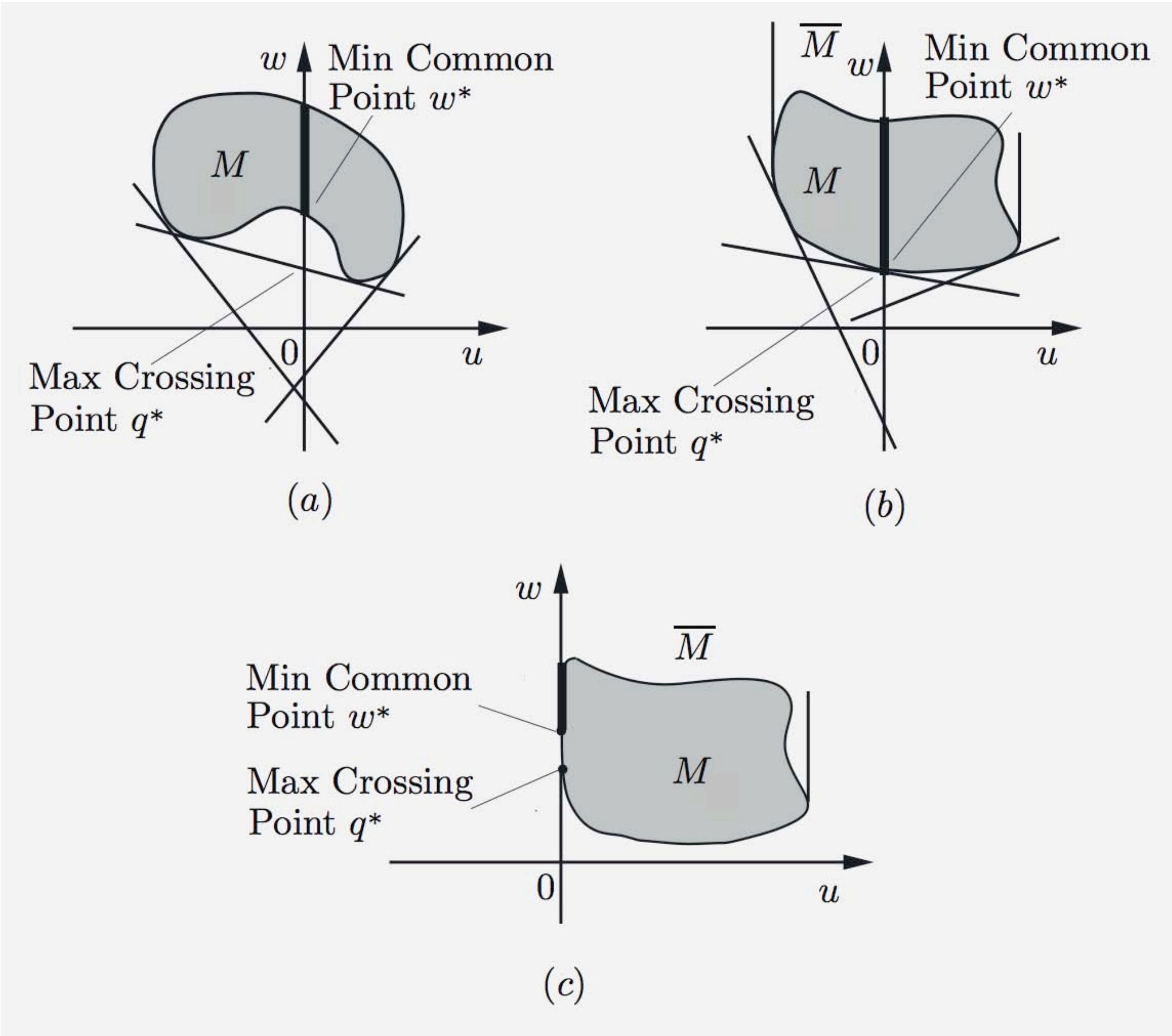
- Despite its elegance, the Fenchel framework is somewhat indirect.

- From duality of set descriptions, to
  - duality of functional descriptions, to
  - duality of problem descriptions.

- A more direct approach:
  - Start with a set, then
  - Define two simple prototype problems dual to each other.

- Avoid functional descriptions (a simpler, less constrained framework).

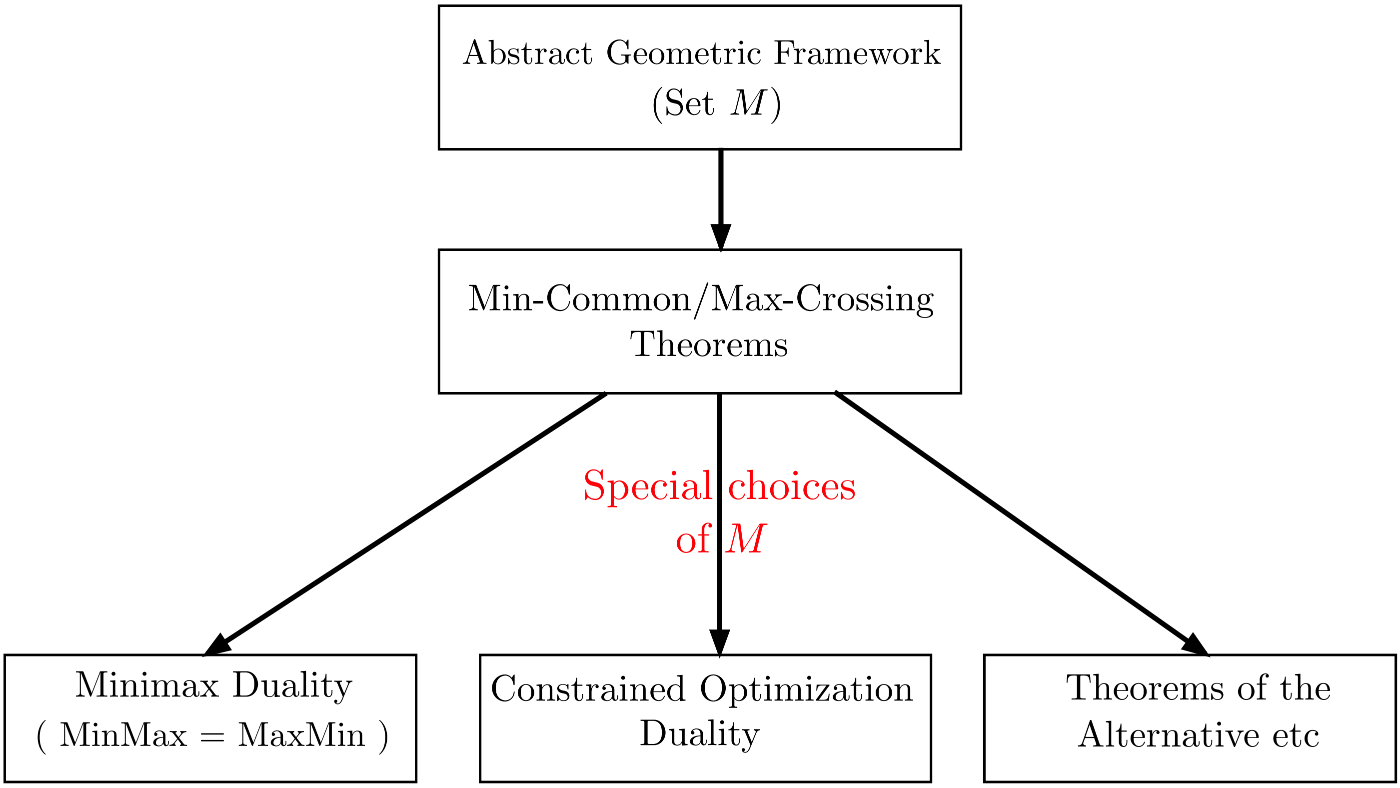
MIN COMMON/MAX CROSSING DUALITY



• All of duality theory and all of (convex/concave) minimax theory can be developed/explained in terms of this one figure.

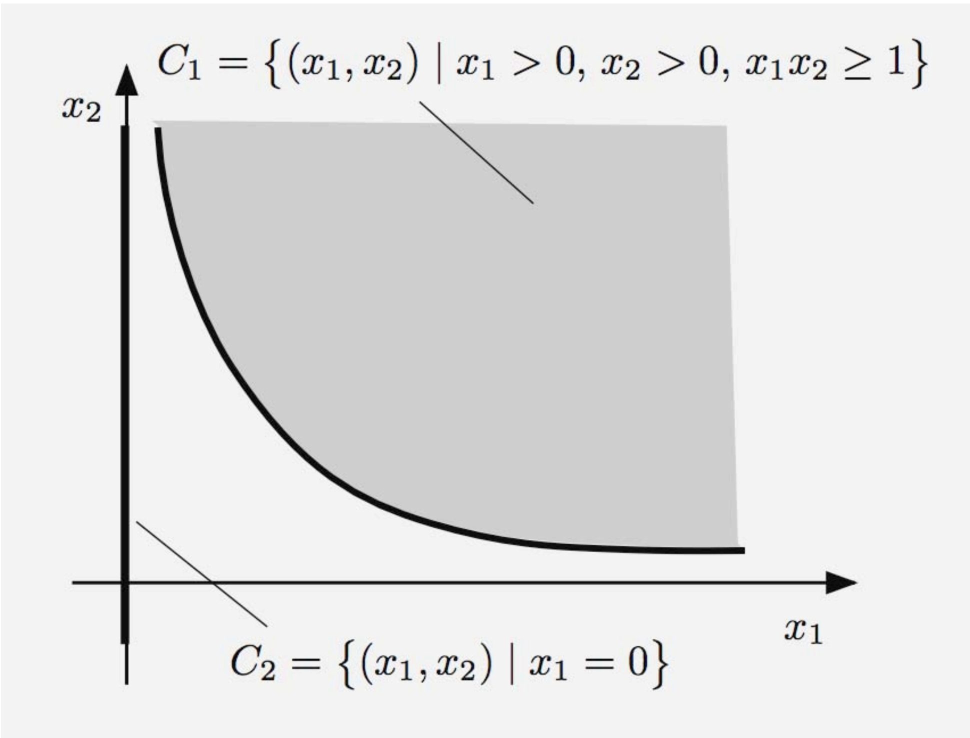
• The machinery of convex analysis is needed to flesh out this figure, and to rule out the exceptional/pathological behavior shown in (c).

ABSTRACT/GENERAL DUALITY ANALYSIS



EXCEPTIONAL BEHAVIOR

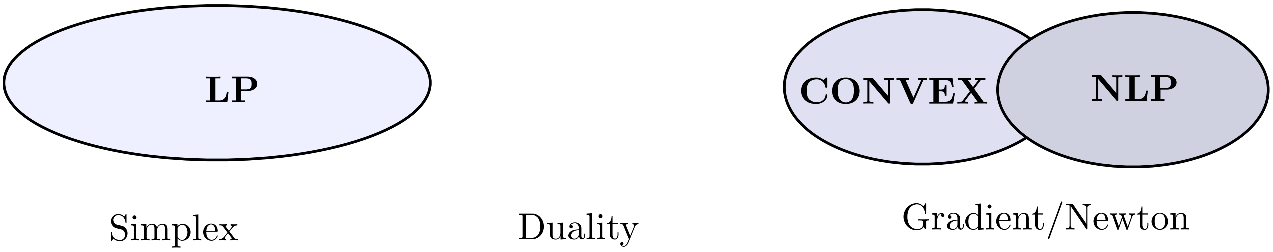
- If convex structure is so favorable, what is the source of exceptional/pathological behavior?
- **Answer:** Some common operations on convex sets do not preserve some basic properties.
- **Example:** A linearly transformed closed convex set need not be closed (contrary to compact and polyhedral sets).
  - Also the vector sum of two closed convex sets need not be closed.



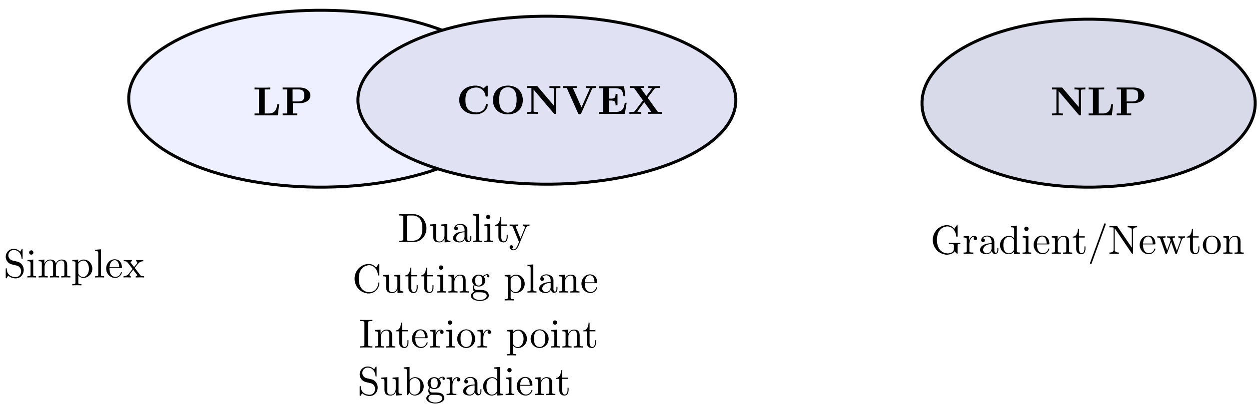
- This is a major reason for the analytical difficulties in convex analysis and pathological behavior in convex optimization (and the favorable character of polyhedral sets).

MODERN VIEW OF CONVEX OPTIMIZATION

- Traditional view: Pre 1990s
  - LPs are solved by simplex method
  - NLPs are solved by gradient/Newton methods
  - Convex programs are special cases of NLPs



- Modern view: Post 1990s
  - LPs are often solved by nonsimplex/convex methods
  - Convex problems are often solved by the same methods as LPs
  - "Key distinction is not Linear-Nonlinear but Convex-Nonconvex" (Rockafellar)



THE RISE OF THE ALGORITHMIC ERA

- Convex programs and LPs connect around
  - Duality
  - Large-scale piecewise linear problems
- Synergy of:
  - Duality
  - Algorithms
  - Applications
- New problem paradigms with rich applications
- Duality-based decomposition
  - Large-scale resource allocation
  - Lagrangian relaxation, discrete optimization
  - Stochastic programming
- Conic programming
  - Robust optimization
  - Semidefinite programming
- Machine learning
  - Support vector machines
  - l_1 regularization/Robust regression/Compressed sensing

METHODOLOGICAL TRENDS

- New methods, renewed interest in old methods.
  - Interior point methods
  - Subgradient/incremental methods
  - Polyhedral approximation/cutting plane methods
  - Regularization/proximal methods
  - Incremental methods

- Renewed emphasis on complexity analysis
  - Nesterov, Nemirovski, and others ...
  - "Optimal algorithms" (e.g., extrapolated gradient methods)

- Emphasis on interesting (often duality-related) large-scale special structures

COURSE OUTLINE

- We will follow closely the textbook
  - Bertsekas, "Convex Optimization Theory," Athena Scientific, 2009, including the on-line Chapter 6 and supplementary material at http://www.athenasc.com/convexduality.html
- Additional book references:
  - Rockafellar, "Convex Analysis," 1970.
  - Boyd and Vanderbergue, "Convex Optimization," Cambridge U. Press, 2004. (On-line at http://www.stanford.edu/~boyd/cvxbook/)
  - Bertsekas, Nedic, and Ozdaglar, "Convex Analysis and Optimization," Ath. Scientific, 2003.
- Topics (the text's design is modular, and the following sequence involves no loss of continuity):
  - **Basic Convexity Concepts:** Sect. 1.1-1.4.
  - **Convexity and Optimization:** Ch. 3.
  - **Hyperplanes & Conjugacy:** Sect. 1.5, 1.6.
  - **Polyhedral Convexity:** Ch. 2.
  - **Geometric Duality Framework:** Ch. 4.
  - **Duality Theory:** Sect. 5.1-5.3.
  - **Subgradients:** Sect. 5.4.
  - **Algorithms:** Ch. 6.

WHAT TO EXPECT FROM THIS COURSE

- Requirements: Homework (25%), midterm (25%), and a term paper (50%)
- We aim:
  - To develop insight and deep understanding of a fundamental optimization topic
  - To treat with mathematical rigor an important branch of methodological research, and to provide an account of the state of the art in the field
  - To get an understanding of the merits, limitations, and characteristics of the rich set of available algorithms
- Mathematical level:
  - Prerequisites are linear algebra (preferably abstract) and real analysis (a course in each)
  - Proofs will matter ... but the rich geometry of the subject helps guide the mathematics
- Applications:
  - They are many and pervasive ... but don't expect much in this course. The book by Boyd and Vandenberghe describes a lot of practical convex optimization models
  - You can do your term paper on an application area

A NOTE ON THESE SLIDES

- These slides are a teaching aid, not a text
- Don't expect a rigorous mathematical development
- The statements of theorems are fairly precise, but the proofs are not
- Many proofs have been omitted or greatly abbreviated
- Figures are meant to convey and enhance understanding of ideas, not to express them precisely
- The omitted proofs and a fuller discussion can be found in the "Convex Optimization Theory" textbook and its supplementary material

6.253 Convex Analysis and Optimization
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