6.252 NONLINEAR PROGRAMMING

LECTURE 15: INTERIOR POINT METHODS

LECTURE OUTLINE

- Barrier and Interior Point Methods
- Linear Programs and the Logarithmic Barrier
- Path Following Using Newton's Method

Inequality constrained problem

minimize $f(x)$

subject to $x \in X, \qquad g_j(x) \leq b_j,\ j = 1, \ldots, r,$

where f and g_j are continuous and X is closed. We assume that the set

$$S = \left\{ x \in X \mid g_j(x) < 0,\ j = 1, \ldots, r \right\}$$

is nonempty and any feasible point is in the closure of S.

BARRIER METHOD

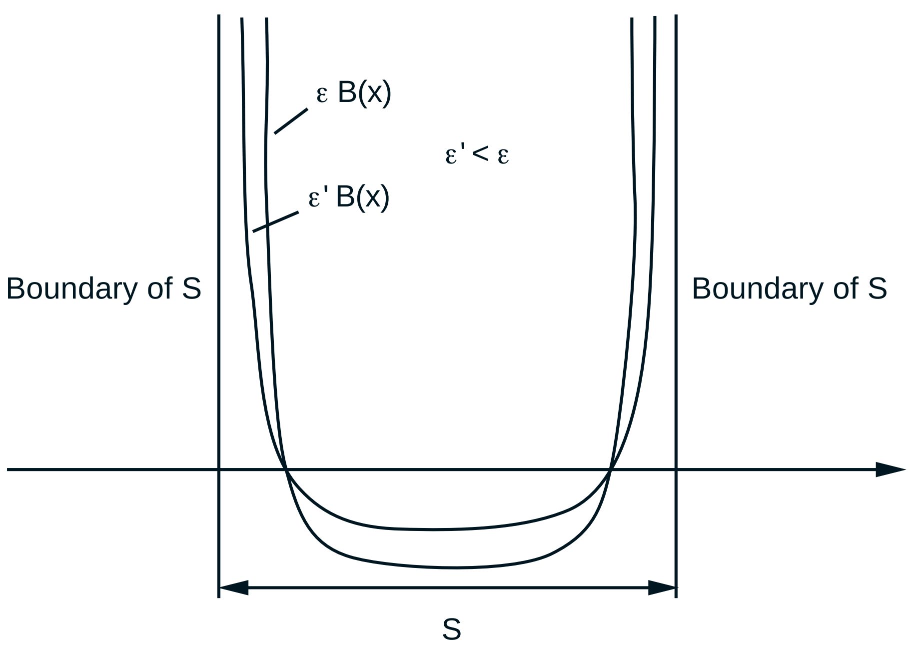
- Consider a *barrier function*, that is continuous and goes to ∞ as any one of the constraints $g_j(x)$ approaches 0 from negative values. Examples:

$$B(x) = -\sum_{j=1}^{r} \ln\Big\{-g_j(x)\Big\}, \qquad B(x) = -\sum_{j=1}^{r} \frac{1}{g_j(x)}.$$

- Barrier Method:

$$x^k = \arg\min_{x \in S}\Big\{f(x) + \epsilon^k B(x)\Big\}, \qquad k = 0, 1, \ldots,$$

where the parameter sequence $\{\epsilon^k\}$ satisfies $0 < \epsilon^{k+1} < \epsilon^k$ for all k and $\epsilon^k \to 0$.



CONVERGENCE

Every limit point of a sequence $\{x^k\}$ generated by a barrier method is a global minimum of the original constrained problem

Proof: Let $\{\overline{x}\}$ be the limit of a subsequence $\{x^k\}_{k \in K}$. Since $x^k \in S$ and X is closed, $\overline{x}$ is feasible for the original problem. If $\overline{x}$ is not a global minimum, there exists a feasible x^* such that $f(x^*) < f(\overline{x})$ and therefore also an interior point $\tilde{x} \in S$ such that $f(\tilde{x}) < f(\overline{x})$. By the definition of x^k, $f(x^k) + \epsilon^k B(x^k) \leq f(\tilde{x}) + \epsilon^k B(\tilde{x})$ for all k, so by taking limit

$$f(\overline{x}) + \liminf_{k \to \infty,\, k \in K} \epsilon^k B(x^k) \leq f(\tilde{x}) < f(\overline{x})$$

Hence $\liminf_{k \to \infty,\, k \in K} \epsilon^k B(x^k) < 0$.

If $\overline{x} \in S$, we have $\lim_{k \to \infty,\, k \in K} \epsilon^k B(x^k) = 0$, while if $\overline{x}$ lies on the boundary of S, we have by assumption $\lim_{k \to \infty,\, k \in K} B(x^k) = \infty$. Thus

$$\liminf_{k \to \infty} \epsilon^k B(x^k) \geq 0,$$

– a contradiction.

LINEAR PROGRAMS/LOGARITHMIC BARRIER

- Apply logarithmic barrier to the linear program

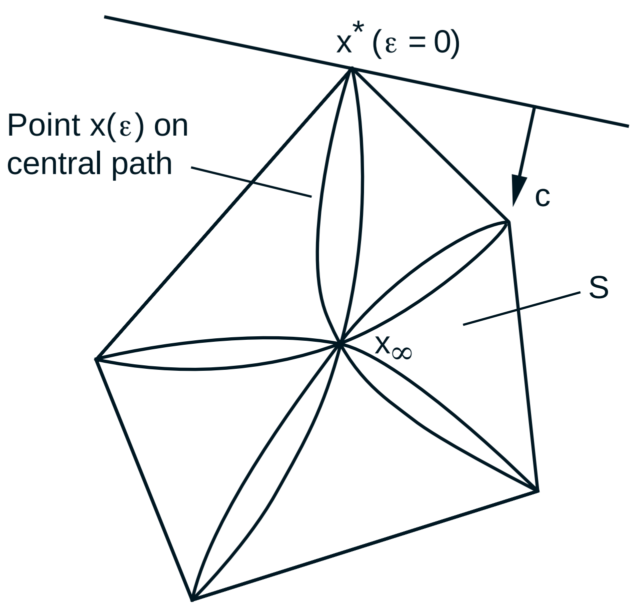
$$\begin{aligned} &\text{minimize } c'x \\ &\text{subject to } Ax = b, \qquad x \geq 0, \end{aligned} \tag{LP}$$

The method finds for various $\epsilon > 0$,

$$x(\epsilon) = \arg\min_{x \in S} F_\epsilon(x) = \arg\min_{x \in S} \left\{ c'x - \epsilon \sum_{i=1}^{n} \ln x_i \right\},$$

where $S = \{x \mid Ax = b,\ x > 0\}$. We assume that S is nonempty and bounded.

- As $\epsilon \to 0$, $x(\epsilon)$ follows the *central path*



All central paths start at the *analytic center*

$$x_\infty = \arg\min_{x \in S} \left\{ -\sum_{i=1}^{n} \ln x_i \right\},$$

and end at optimal solutions of (LP).

PATH FOLLOWING W/ NEWTON'S METHOD

- Newton's method for minimizing F_ϵ:

$$\tilde{x} = x + \alpha(\overline{x} - x),$$

where $\overline{x}$ is the pure Newton iterate

$$\overline{x} = \arg\min_{Az=b} \left\{ \nabla F_\epsilon(x)'(z - x) + \tfrac{1}{2}(z - x)'\nabla^2 F_\epsilon(x)(z - x) \right\}$$

- By straightforward calculation

$$\overline{x} = x - Xq(x, \epsilon),$$

$$q(x, \epsilon) = \frac{Xz}{\epsilon} - e, \quad e = (1 \ldots 1)', \quad z = c - A'\lambda,$$

$$\lambda = (AX^2A')^{-1}AX\Big(Xc - \epsilon e\Big),$$

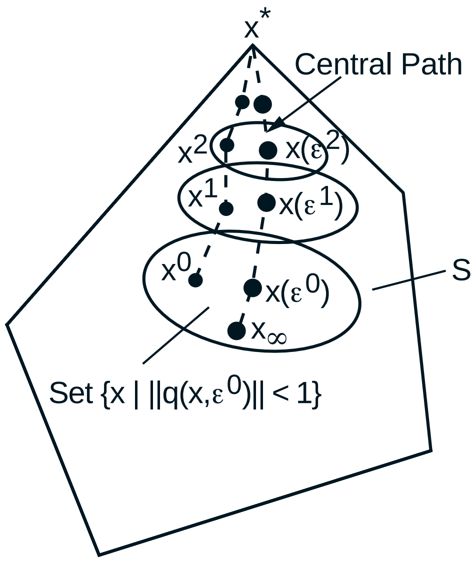
and X is the diagonal matrix with x_i, $i = 1, \ldots, n$ along the diagonal.

- View $q(x, \epsilon)$ as the Newton increment $(x - \overline{x})$ transformed by X^{-1} that maps x into e.

- Consider $\|q(x, \epsilon)\|$ as a *proximity measure* of the current point to the point $x(\epsilon)$ on the central path.

KEY RESULTS

- It is sufficient to minimize F_ϵ approximately, up to where $\|q(x,\epsilon)\| < 1$.



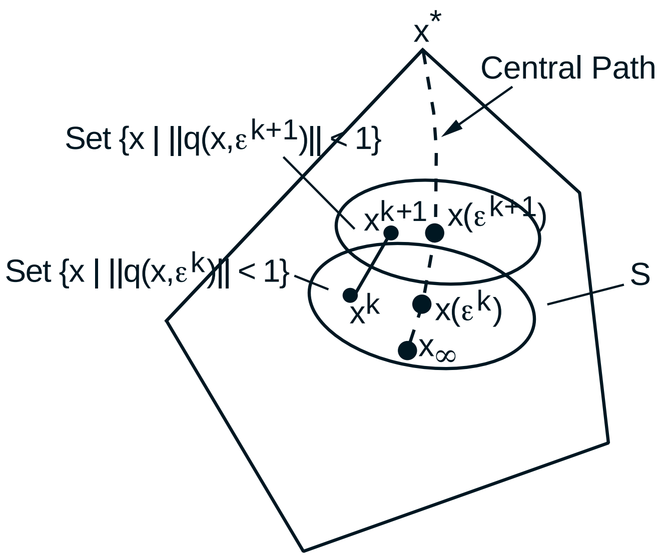
If $x > 0$, $Ax = b$, and $\|q(x,\epsilon)\| < 1$, then

$$c'x - \min_{Ay=b,\, y\geq 0} c'y \leq \epsilon\Big(n + \sqrt{n}\Big).$$

- The "termination set" $\big\{x \mid \|q(x,\epsilon)\| < 1\big\}$ is part of the region of quadratic convergence of the pure form of Newton's method. In particular, if $\|q(x,\epsilon)\| < 1$, then the pure Newton iterate $\overline{x} = x - Xq(x,\epsilon)$ is an interior point, that is, $\overline{x} \in S$. Furthermore, we have $\|q(\overline{x},\epsilon)\| < 1$ and in fact

$$\|q(\overline{x},\epsilon)\| \leq \|q(x,\epsilon)\|^2.$$

SHORT STEP METHODS



Following approximately the central path by using a single Newton step for each ϵ^k. If ϵ^k is close to ϵ^{k+1} and x^k is close to the central path, one expects that x^{k+1} obtained from x^k by a single pure Newton step will also be close to the central path.

Proposition Let $x > 0$, $Ax = b$, and suppose that for some $\gamma < 1$ we have $\|q(x, \epsilon)\| \leq \gamma$. Then if $\overline{\epsilon} = (1 - \delta n^{-1/2})\epsilon$ for some $\delta > 0$,

$$\|q(\overline{x}, \overline{\epsilon})\| \leq \frac{\gamma^2 + \delta}{1 - \delta n^{-1/2}}.$$

In particular, if

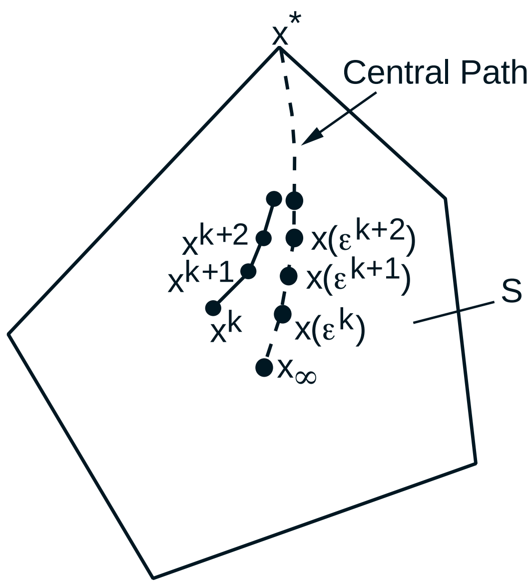
$$\delta \leq \gamma(1 - \gamma)(1 + \gamma)^{-1},$$

we have $\|q(\overline{x}, \overline{\epsilon})\| \leq \gamma$.

- Can be used to establish nice complexity results; but ϵ must be reduced VERY slowly.

LONG STEP METHODS

- Main features:
  - Decrease ϵ faster than dictated by complexity analysis.
  - Require more than one Newton step per (approximate) minimization.
  - Use line search as in unconstrained Newton's method.
  - Require much smaller number of (approximate) minimizations.



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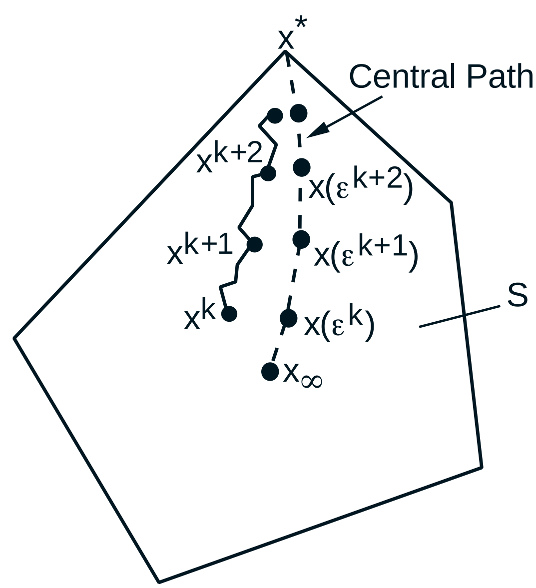

(b)

- The methodology generalizes to quadratic programming and convex programming.