15.093 Optimization Methods

Lecture 10: Network Optimization
The Network Simplex Algorithm

Network Optimization — Why do we care?

- Networks and associated optimization problems constitute reoccurring structures in many real-world applications.
- The network structure often leads to additional insight and improved understanding.
- It also enable us to design more efficient algorithms.

Outline — Today's Lecture

- The Simplex Algorithm: A Reminder
- The Network Simplex: A Combinatorial View
- The Network Simplex: An Animated View
- The Network Simplex: An Algebraic View

The Simplex Algorithm — A Reminder — The Problem...

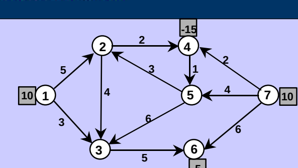
$$\begin{aligned} \min\ & c'x \\ \text{s.t.}\ & Ax = b \\ & x \geq 0 \end{aligned}$$

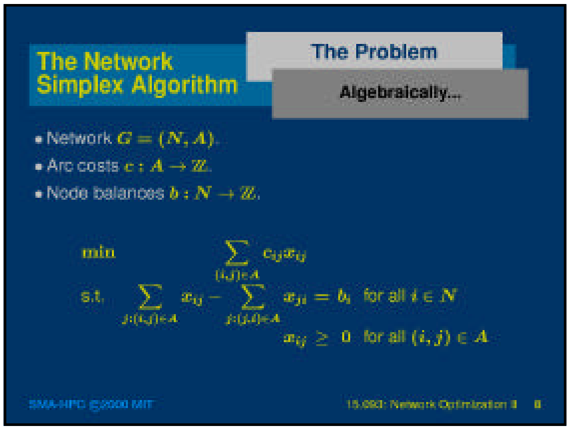
The Simplex Algorithm — A Reminder — The Algorithm

1. Start with basis $B = [A_{B(1)}, \ldots, A_{B(m)}]$ and BFS x.
2. Compute $\bar{c}_j = c_j - c'_B B^{-1} A_j$.
   - If $\bar{c}_j \geq 0$; x optimal; stop.
   - Select j such that $\bar{c}_j < 0$.
3. Compute $u = B^{-1} A_j$. $\theta^* = \min_{1 \leq i \leq m, u_i > 0} \frac{x_{B(i)}}{u_i} = \frac{x_{B(\ell)}}{u_\ell}$.
4. Form a new basis by replacing $A_{B(\ell)}$ with A_j.
5. $y_j = \theta^*$; $y_{B(i)} = x_{B(i)} - \theta^* u_i$.

The Network Simplex Algorithm — The Problem — Combinatorially...

Determine a least cost shipment of a commodity through a network in order to satisfy demands at certain nodes from available supplies at other nodes. Arcs have costs associated with them.




The Network Simplex Algorithm
The Problem
Algebraically...
• Network $G = (N, A)$.
• Arc costs $c : A \to \mathbb{Z}$.
• Node balances $b : N \to \mathbb{Z}$.
min $\sum_{(i,j) \in A} c_{ij} x_{ij}$
s.t. $\sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(j,i) \in A} x_{ji} = b_i$ for all $i \in N$
$x_{ij} \geq 0$ for all $(i,j) \in A$
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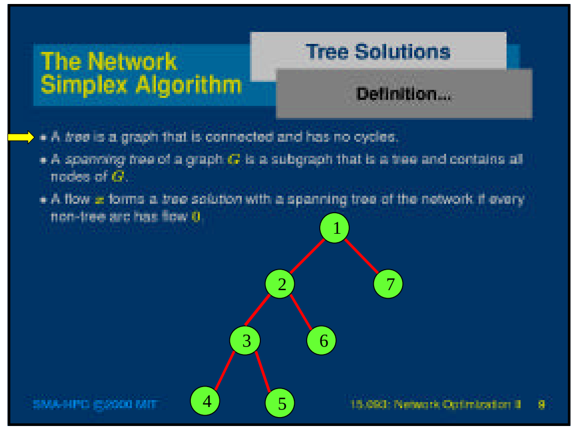
The Network Simplex Algorithm
Tree Solutions
Definition...
• A *tree* is a graph that is connected and has no cycles.
• A *spanning tree* of a graph G is a subgraph that is a tree and contains all nodes of G.
• A flow x forms a *tree solution* with a spanning tree of the network if every non-tree arc has flow 0.
1
2
7
3
6
4
5
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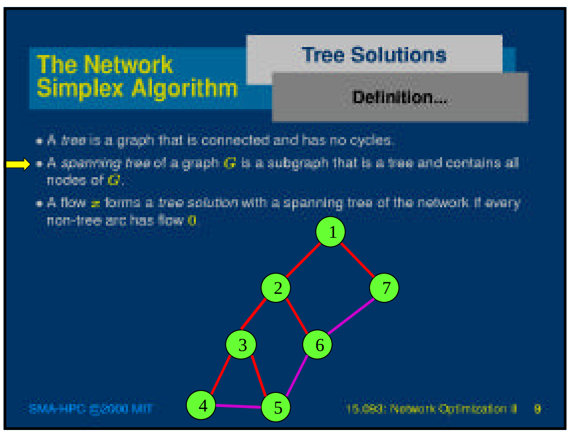
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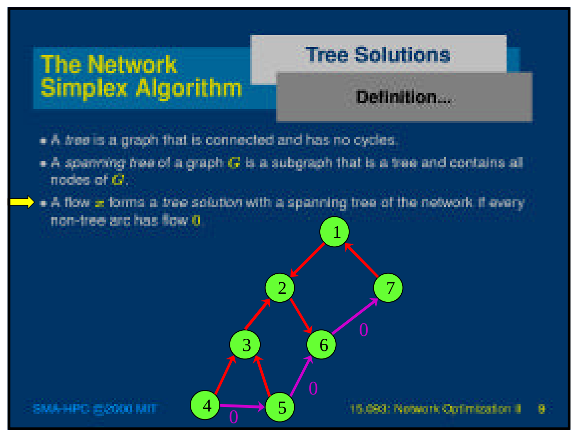
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2
7
0
3
6
0
4
0
5
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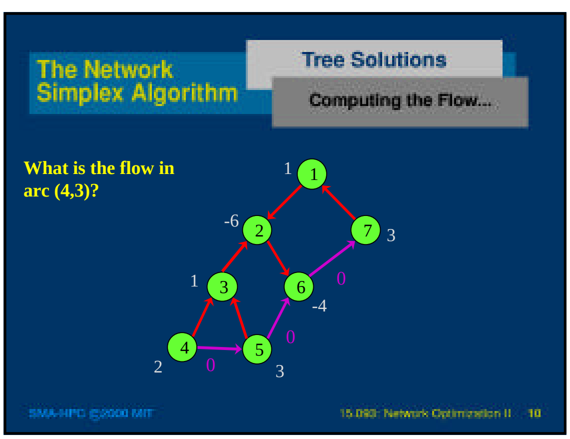
The Network Simplex Algorithm
Tree Solutions
Computing the Flow...
What is the flow in arc (4,3)?
1
1
-6
2
7
3
1
3
6
0
-4
0
4
5
2
0
3
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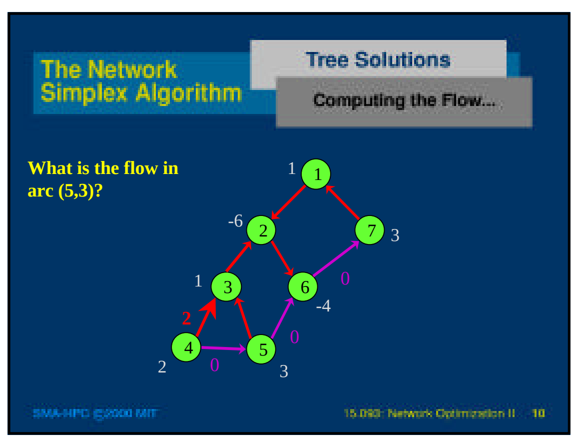
The Network Simplex Algorithm
Tree Solutions
Computing the Flow...
What is the flow in arc (5,3)?
1
1
-6
2
7
3
1
3
6
0
2
-4
0
4
5
2
0
3
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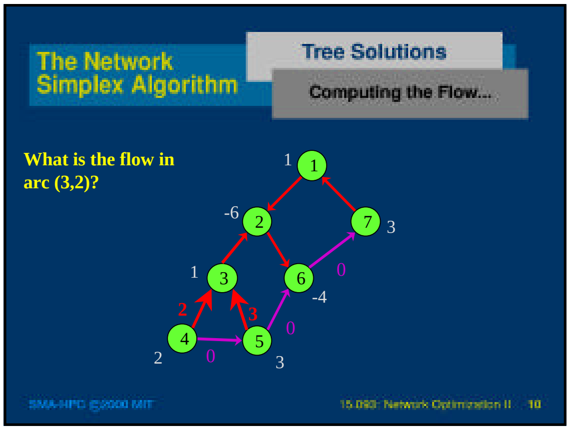
The Network Simplex Algorithm
Tree Solutions
Computing the Flow...
What is the flow in arc (3,2)?
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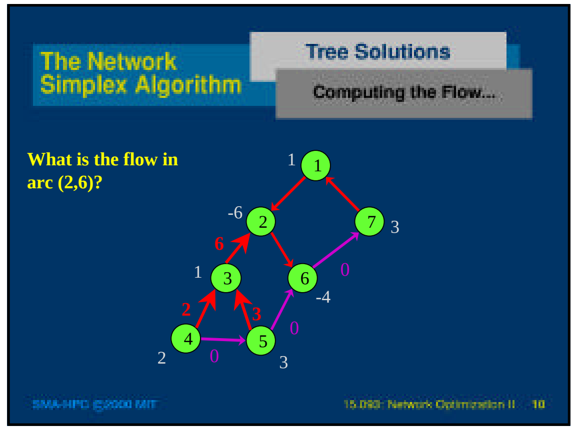
The Network Simplex Algorithm
Tree Solutions
Computing the Flow...
What is the flow in arc (2,6)?
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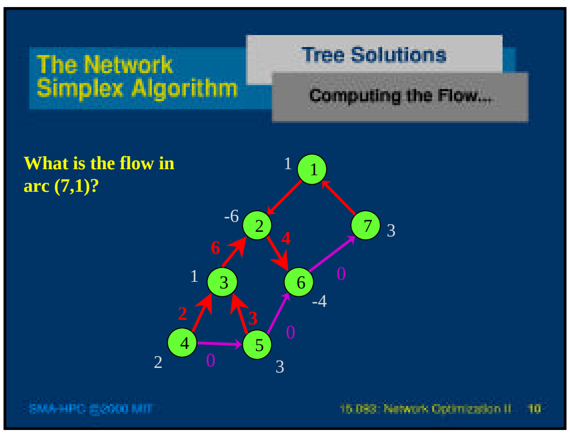
The Network Simplex Algorithm
Tree Solutions
Computing the Flow...
What is the flow in arc (7,1)?
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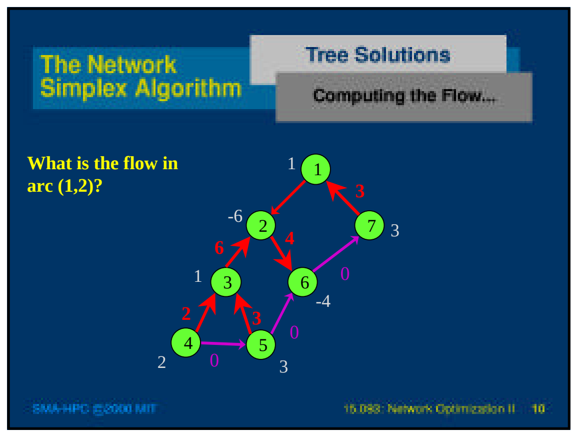
The Network Simplex Algorithm
Tree Solutions
Computing the Flow...
What is the flow in arc (1,2)?
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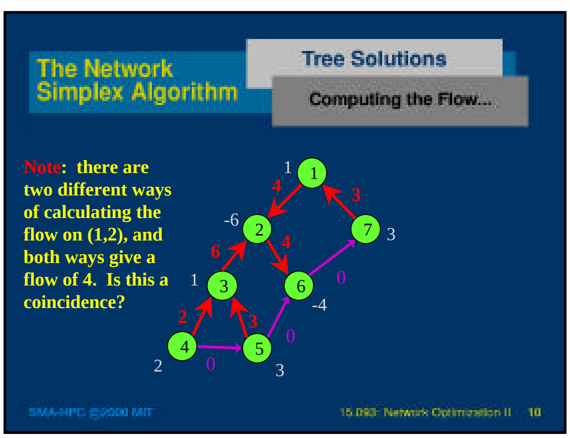
The Network Simplex Algorithm
Tree Solutions
Computing the Flow...
Note: there are two different ways of calculating the flow on (1,2), and both ways give a flow of 4. Is this a coincidence?
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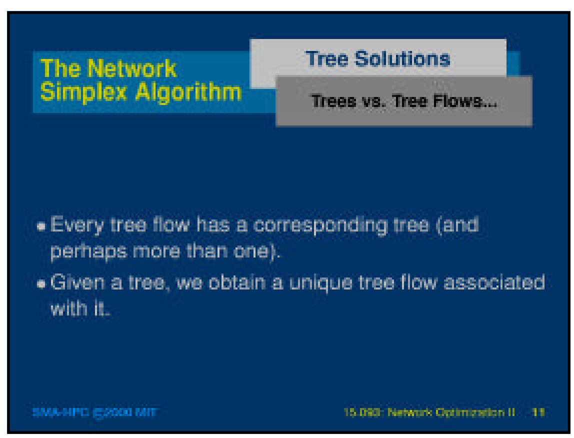
The Network Simplex Algorithm
Tree Solutions
Trees vs. Tree Flows...
• Every tree flow has a corresponding tree (and perhaps more than one).
• Given a tree, we obtain a unique tree flow associated with it.
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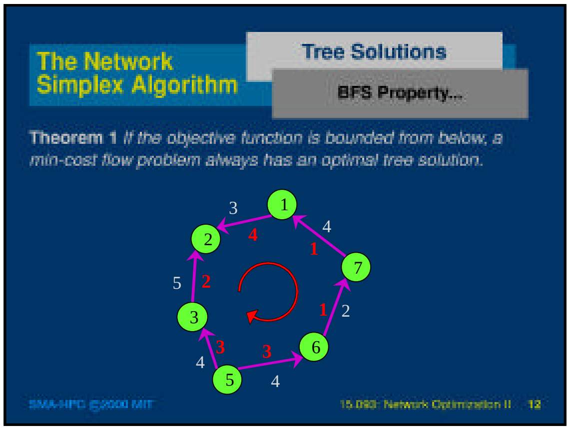
The Network Simplex Algorithm
Tree Solutions
BFS Property...
Theorem 1 *If the objective function is bounded from below, a min-cost flow problem always has an optimal tree solution.*
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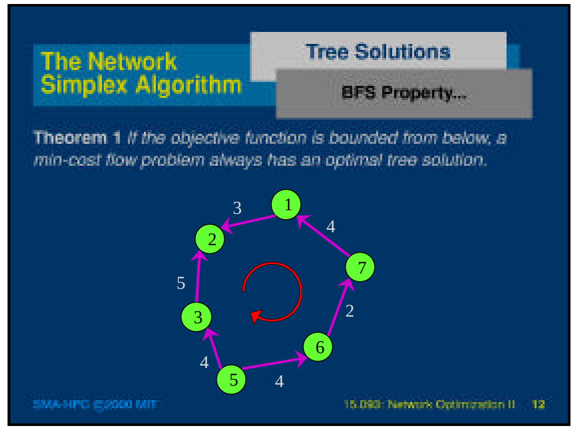
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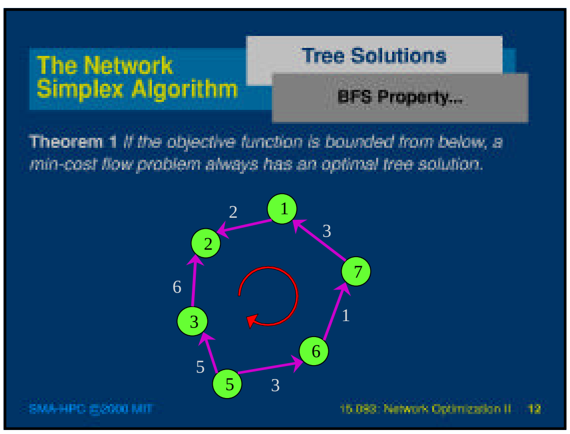
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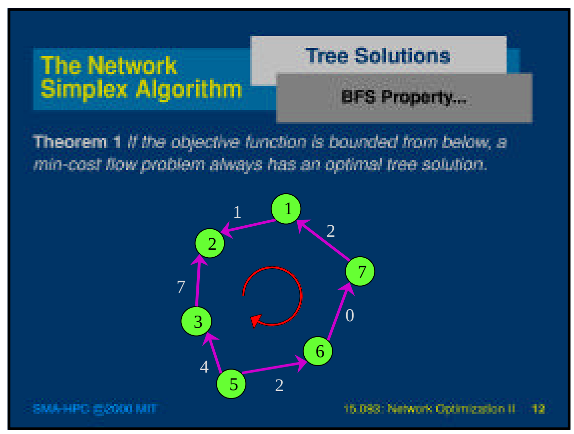
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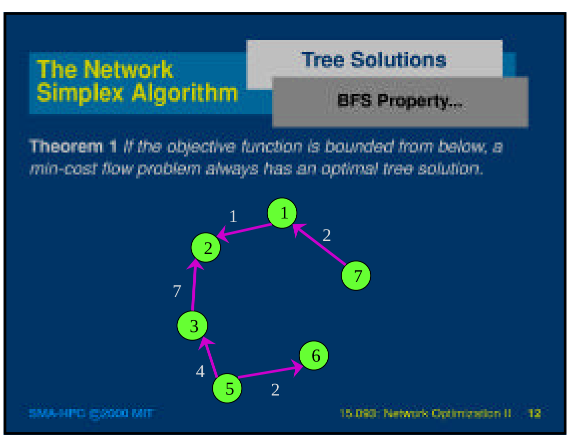
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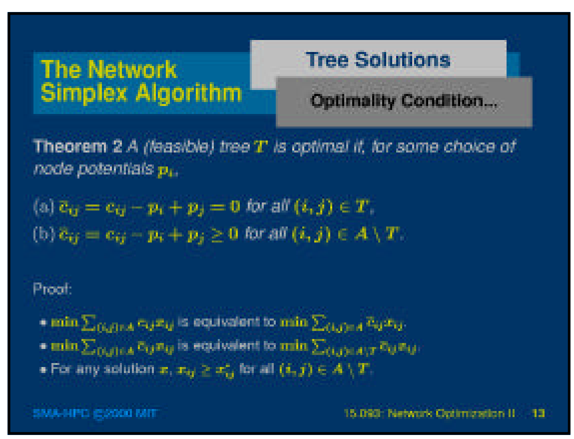
The Network Simplex Algorithm
Tree Solutions
Optimality Condition...
Theorem 2 *A (feasible) tree T is optimal if, for some choice of node potentials p_i,*
(a) $\bar{c}_{ij} = c_{ij} - p_i + p_j = 0$ *for all* $(i,j) \in T$,
(b) $\bar{c}_{ij} = c_{ij} - p_i + p_j \geq 0$ *for all* $(i,j) \in A \setminus T$.
Proof:
• $\min \sum_{(i,j)\in A} c_{ij}x_{ij}$ is equivalent to $\min \sum_{(i,j)\in A} \bar{c}_{ij}x_{ij}$.
• $\min \sum_{(i,j)\in A} \bar{c}_{ij}x_{ij}$ is equivalent to $\min \sum_{(i,j)\in A\setminus T} \bar{c}_{ij}x_{ij}$.
• For any solution x, $x_{ij} \geq x^*_{ij}$ for all $(i,j) \in A \setminus T$.
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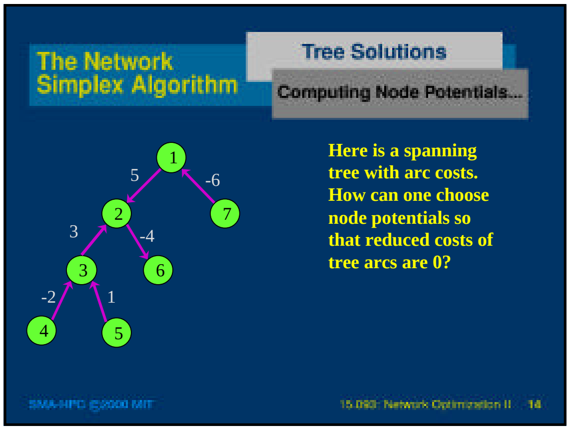
The Network Simplex Algorithm
Tree Solutions
Computing Node Potentials...
Here is a spanning tree with arc costs. How can one choose node potentials so that reduced costs of tree arcs are 0?
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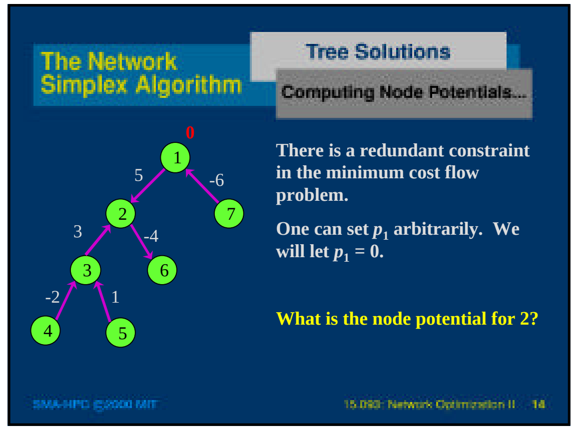
The Network Simplex Algorithm
Tree Solutions
Computing Node Potentials...
There is a redundant constraint in the minimum cost flow problem.
One can set p_1 arbitrarily. We will let $p_1 = 0$.
What is the node potential for 2?
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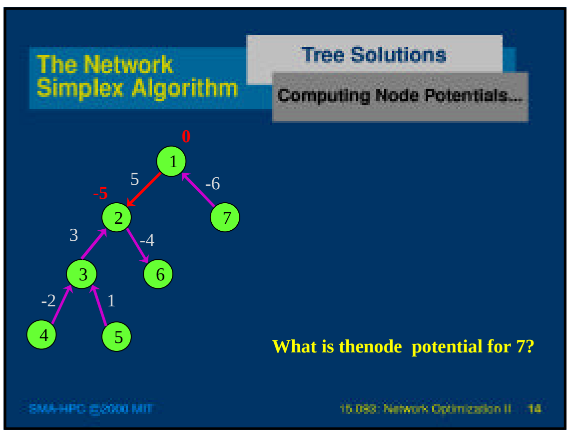
The Network Simplex Algorithm
Tree Solutions
Computing Node Potentials...
What is thenode potential for 7?
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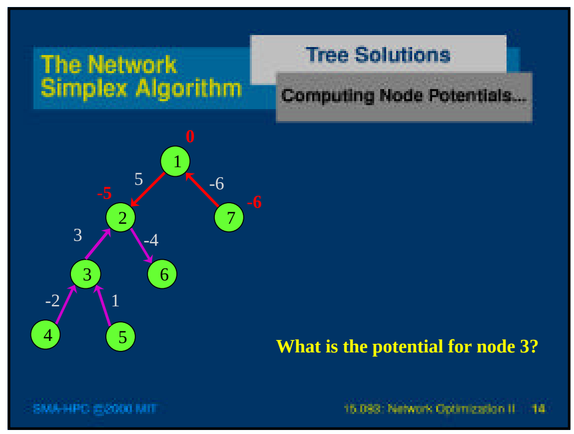
The Network Simplex Algorithm
Tree Solutions
Computing Node Potentials...
What is the potential for node 3?
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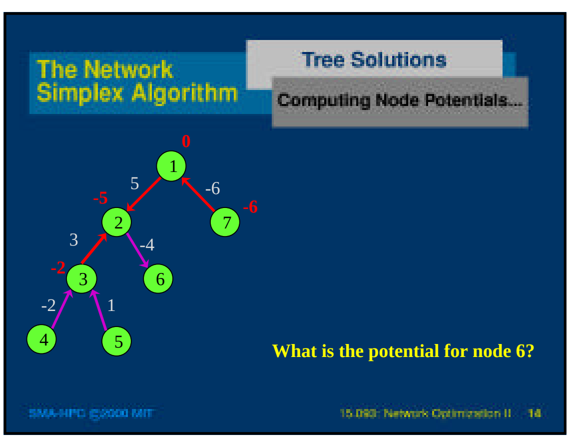
The Network Simplex Algorithm
Tree Solutions
Computing Node Potentials...
What is the potential for node 6?
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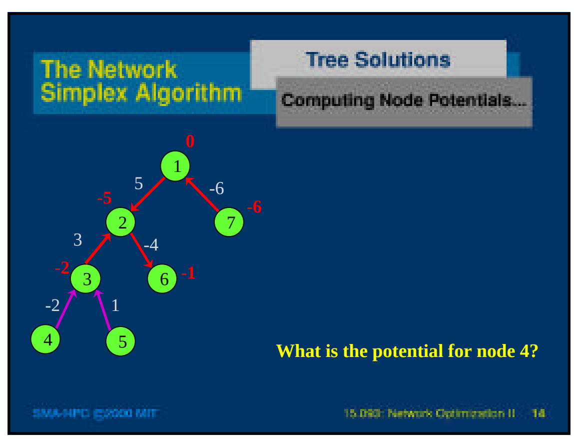
The Network Simplex Algorithm
Tree Solutions
Computing Node Potentials...
What is the potential for node 4?
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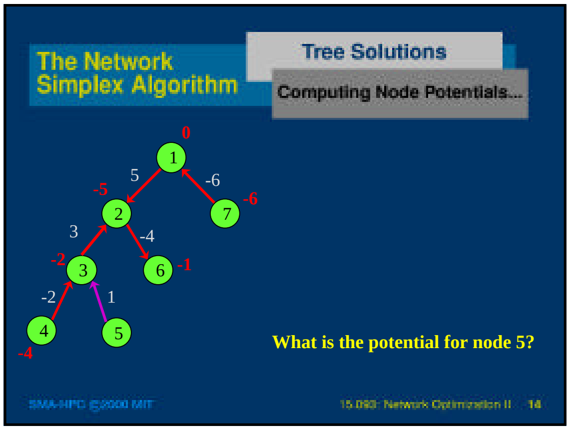
The Network Simplex Algorithm
Tree Solutions
Computing Node Potentials...
What is the potential for node 5?
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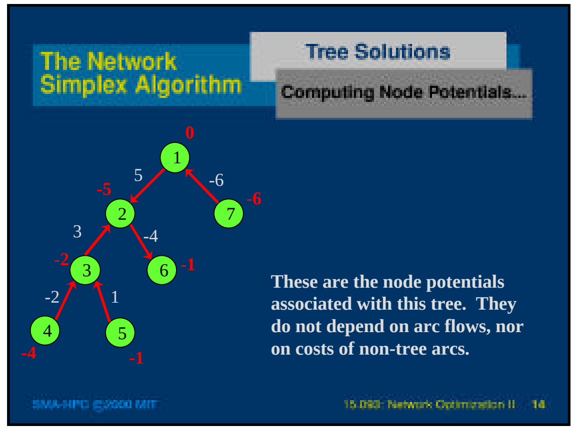
The Network Simplex Algorithm
Tree Solutions
Computing Node Potentials...
These are the node potentials associated with this tree. They do not depend on arc flows, nor on costs of non-tree arcs.
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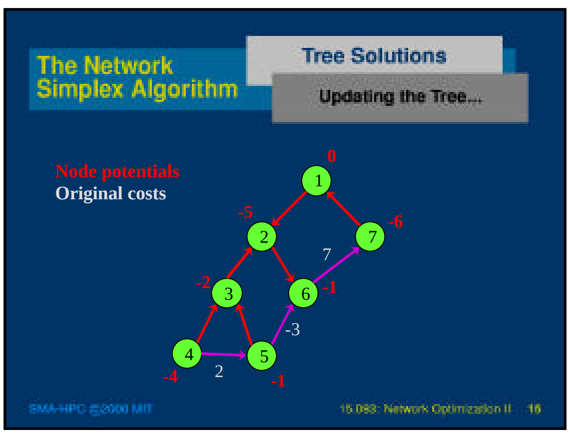
The Network Simplex Algorithm
Tree Solutions
Updating the Tree...
Node potentials
Original costs
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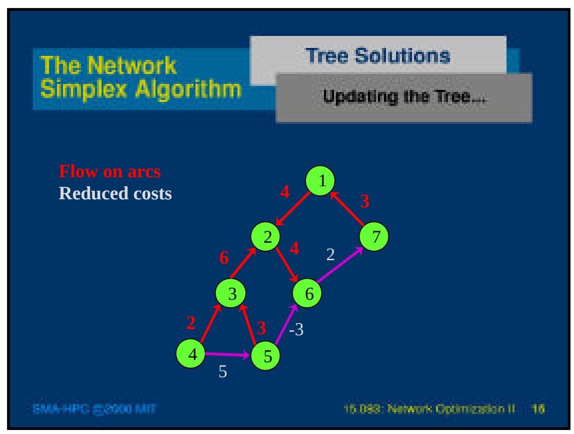
The Network Simplex Algorithm
Tree Solutions
Updating the Tree...
Flow on arcs
Reduced costs
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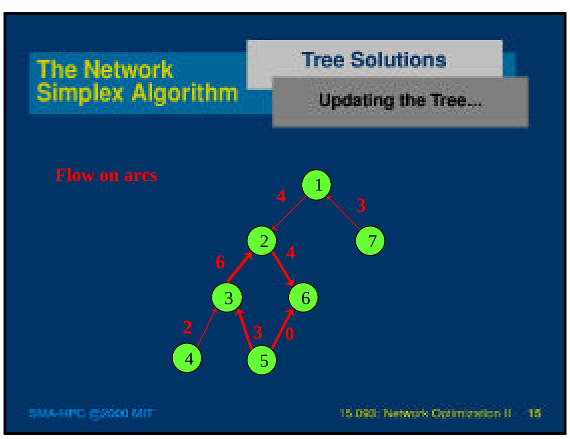
The Network Simplex Algorithm
Tree Solutions
Updating the Tree...
Flow on arcs
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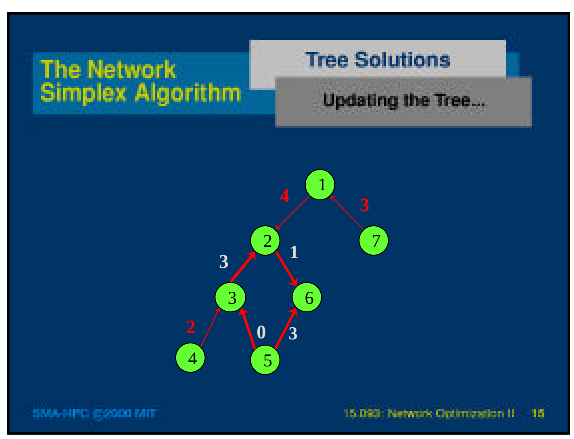
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Tree Solutions
Updating the Tree...
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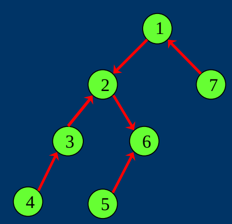
The Network Simplex Algorithm

Tree Solutions

Updating the Tree...



The Network Simplex Algorithm

Overview of the Algorithm

1. Determine an initial feasible tree T. Compute flow x and node potentials p associated with T.
2. Calculate $\bar{c}_{ij} = c_{ij} - p_i + p_j$ for $(i, j) \notin T$.
   - If $\bar{c} \geq \mathbf{0}$, x optimal; stop.
   - Select (i, j) with $\bar{c}_{ij} < 0$.
3. Add (i, j) to T creating a unique cycle C. Send a maximum flow around C while maintaining feasibility. Suppose the exiting arc is (k, ℓ).
4. $T := (T \setminus (k, \ell)) \cup (i, j)$.

Min-Cost Flow

Integrality

Our reasoning has two important and far-reaching implications:

- There always exists an integer optimal flow (if node balances b_i are integer).
- There always exist optimal integer node potentials (if arc costs c_{ij} are integer).

The Network Simplex Algorithm

An Animation

The Network Simplex Algorithm

The Algebraic View

- Bases and trees.
- Dual variables and node potentials.
- Changing bases and updating trees.
- Optimality testing.

The Network Simplex Algorithm

The Algebraic View

Bases vs. Trees...

The constraint matrix A of the min-cost flow problem is the node-arc incidence matrix of the underlying network.

	$(1,2)$	$(2,6)$	$(3,2)$	$(4,3)$	$(4,5)$	$(5,3)$	$(5,6)$	$(6,7)$	$(7,1)$
1	$+1$	0	0	0	0	0	0	0	-1
2	-1	$+1$	-1	0	0	0	0	0	0
3	0	0	$+1$	-1	0	-1	0	0	0
4	0	0	0	$+1$	$+1$	0	0	0	0
5	0	0	0	0	-1	$+1$	$+1$	0	0
6	0	-1	0	0	0	0	-1	$+1$	0
7	0	0	0	0	0	0	0	-1	$+1$

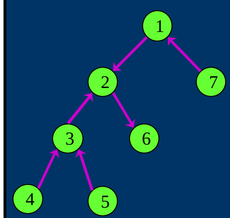
The rows of A are linearly dependent.

The Network Simplex Algorithm

The Algebraic View

...Bases vs. Trees...

Let B be the submatrix corresponding to the tree



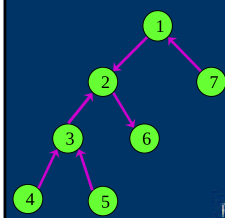
	(1,2)	(2,6)	(3,2)	(4,3)	(5,3)	(7,1)
1	+1	0	0	0	0	−1
2	−1	+1	−1	0	0	0
3	0	0	+1	−1	−1	0
4	0	0	0	+1	0	0
5	0	0	0	0	+1	0
6	0	−1	0	0	0	0
7	0	0	0	0	0	+1

The Network Simplex Algorithm

The Algebraic View

...Bases vs. Trees...

Let B be the submatrix corresponding to the tree



	(1,2)	(2,6)	(3,2)	(4,3)	(5,3)	(7,1)
4	0	0	0	+1	0	0
5	0	0	0	0	+1	0
6	0	−1	0	0	0	0
7	0	0	0	0	0	+1
3	0	0	+1	−1	−1	0
2	−1	+1	−1	0	0	0
1	+1	0	0	0	0	−1

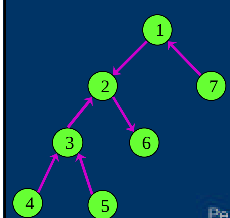
Permuting Rows

The Network Simplex Algorithm

The Algebraic View

...Bases vs. Trees...

Let B be the submatrix corresponding to the tree



	(4,3)	(5,3)	(2,6)	(7,1)	(3,2)	(1,2)
4	+1	0	0	0	0	0
5	0	+1	0	0	0	0
6	0	0	−1	0	0	0
7	0	0	0	+1	0	0
3	−1	−1	0	0	+1	0
2	0	0	+1	0	−1	−1
1	0	0	0	−1	0	+1

Permuting Columns

The Network Simplex Algorithm

The Algebraic View

...Bases vs. Trees...

Corollary 1

(a) *The matrix A has rank $n - 1$.*

(b) *Every tree solution is a basic solution.*

The Network Simplex Algorithm

The Algebraic View

...Bases vs. Trees...

Theorem 3 *Every tree defines a basis and, conversely, every basis defines a tree.*

Suppose the graph defined by a basis contains a cycle $1 - 2 - 3 - 4 - 5 - 6$:

	(1,2)	(2,3)	(4,3)	(5,4)	(5,6)	(1,6)
1	+1	0	0	0	0	+1
2	−1	+1	0	0	0	0
3	0	−1	−1	0	0	0
4	0	0	+1	−1	0	0
5	0	0	0	+1	+1	0
6	0	0	0	0	−1	−1

The Network Simplex Algorithm

The Algebraic View

Dual Variables vs. Node Potentials...

Remember, the simplex algorithm computes the dual variables p as the solution to $p' B = c_B'$

$$(p_4, p_5, p_6, p_7, p_3, p_2) \begin{pmatrix} +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 \\ -1 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & -1 & -1 \end{pmatrix}$$

$$= (c_{43}, c_{53}, c_{26}, c_{71}, c_{32}, c_{12})$$

Hence, $p_2 = -c_{12}$, $p_3 = c_{32} - p_2$, $p_7 = c_{71}, \ldots$

The Network Simplex Algorithm

The Algebraic View

Optimality Testing...

Remember, the simplex algorithm computes the reduced costs $\bar{c}$ as $\bar{c}_{ij} = c_{ij} - p' A_{ij}$.

	(1,2)	(2,6)	(3,2)	(4,3)	(4,5)	(5,3)	(5,6)	(6,7)	(7,1)
1	+1	0	0	0	0	0	0	0	−1
2	−1	+1	−1	0	0	0	0	0	0
3	0	0	+1	−1	0	−1	0	0	0
4	0	0	0	+1	+1	0	0	0	0
5	0	0	0	0	−1	+1	+1	0	0
6	0	−1	0	0	0	0	−1	+1	0
7	0	0	0	0	0	0	0	−1	+1

Therefore, $\bar{c}_{ij} = c_{ij} - p_i + p_j$.

The Network Simplex Algorithm

Summary

- The network simplex algorithm is extremely fast in practice.
- Relying on network data structures, rather than matrix algebra, causes the speedups. It leads to simple rules for selecting the entering and exiting variables.
- Running time per pivot:
  - arcs scanned to identify an entering arc,
  - arcs scanned of the basic cycle,
  - nodes of the subtree.
- A good pivot rule can dramatically reduce running time in practice.