15.081J/6.251J Introduction to Mathematical Programming

Lecture 9: Duality Theory II

1 Outline

SLIDE 1

- Strict complementary slackness
- Geometry of duality
- The dual simplex algorithm
- Duality and degeneracy

2 Strict Complementary Slackness

SLIDE 2

Assume that both problems have an optimal solution:

$$\begin{array}{rrcl} \min & \boldsymbol{c}'\boldsymbol{x} & & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} & \geq & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & \boldsymbol{0}, \end{array} \qquad \begin{array}{rrcl} \max & \boldsymbol{p}'\boldsymbol{b} & & \\ \text{s.t.} & \boldsymbol{p}'\boldsymbol{A} & \leq & \boldsymbol{c}' \\ & \boldsymbol{p} & \geq & \boldsymbol{0}. \end{array}$$

There exist optimal solutions to the primal and to the dual that satisfy

- For every j, either $x_j > 0$ or $\boldsymbol{p}'\boldsymbol{A_j} < c_j$.
- For every i, we have either $\boldsymbol{a_i'x} > b_i$ or $p_i > 0$.

2.1 Example

SLIDE 3

$$\begin{array}{rr} \min & 5x_1 + 5x_2 \\ \text{s.t.} & x_1 + x_2 \geq 2 \\ & 2x_1 - x_2 \geq 0 \\ & x_1, x_2 \geq 0. \end{array}$$

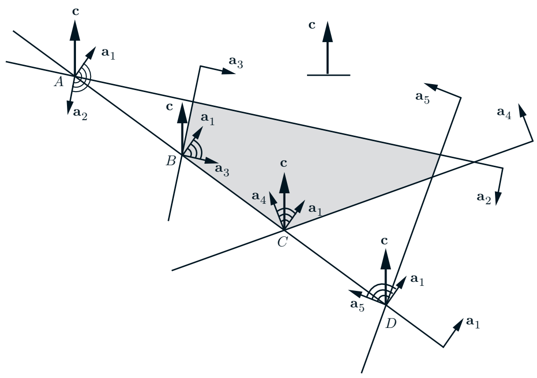
- Is $(2/3, 4/3)$ strictly complementary?
- Which are all the strictly complementary solutions?

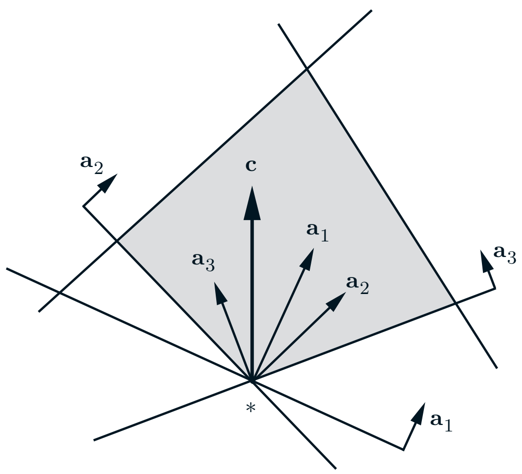
3 The Geometry of Duality

SLIDE 4

$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i'\boldsymbol{x} \geq b_i, \qquad i = 1, \ldots, m \end{array}$$

$$\begin{array}{ll} \max & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} & \displaystyle\sum_{i=1}^{m} p_i \boldsymbol{a}_i = \boldsymbol{c} \\ & \boldsymbol{p} \geq \boldsymbol{0} \end{array}$$


c
a1
a2
A
a3
c
a1
B
a3
c
a4
a1
C
a5
a4
a2
c
a1
a5
D
a1


a2
c
a1
a3
a2
a3
*
a1

4 Dual Simplex Algorithm

4.1 Motivation

SLIDE 5

- In simplex method $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$
- **Primal optimality condition**

$$\boldsymbol{c}' - \boldsymbol{c}'_B \boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}'$$

  same as **dual feasibility**

- Simplex is a **primal algorithm**: maintains **primal feasibility** and works towards **dual feasibility**
- **Dual algorithm**: maintains **dual feasibility** and works towards **primal feasibility**

SLIDE 6

$-\boldsymbol{c}'_B \boldsymbol{x}_B$	$\bar{c}_1$	$\ldots$	$\bar{c}_n$
$x_{B(1)}$	$\vert$		$\vert$
$\vdots$	$\boldsymbol{B}^{-1}\boldsymbol{A}_1$	$\ldots$	$\boldsymbol{B}^{-1}\boldsymbol{A}_n$
$x_{B(m)}$	$\vert$		$\vert$

- Do not require $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$
- Require $\bar{\boldsymbol{c}} \geq \boldsymbol{0}$ (dual feasibility)
- Dual cost is

$$\boldsymbol{p}'\boldsymbol{b} = \boldsymbol{c}'_B \boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{c}'_B \boldsymbol{x}_B$$

- If $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$ then both dual feasibility and primal feasibility, and also same cost $\Rightarrow$ **optimality**
- Otherwise, change basis

4.2 An iteration

SLIDE 7

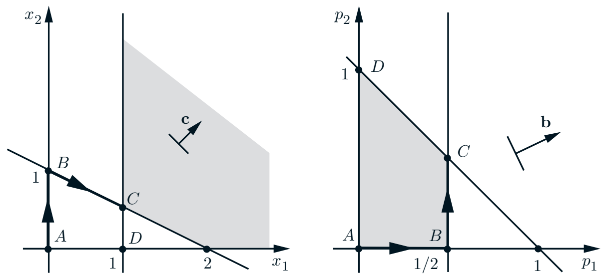
1. Start with basis matrix $\boldsymbol{B}$ and all reduced costs ≥ 0.
2. If $\boldsymbol{B}^{-1}\boldsymbol{b} \geq 0$ optimal solution found; else, choose l s.t. $x_{B(l)} < 0$.
3. Consider the lth row (pivot row) $x_{B(l)}, v_1, \ldots, v_n$. If $\forall i$ $v_i \geq 0$ then dual optimal cost $= +\infty$ and algorithm terminates.

SLIDE 8

4. Else, let j s.t.

$$\frac{\bar{c}_j}{|v_j|} = \min_{\{i | v_i < 0\}} \frac{\bar{c}_i}{|v_i|}$$

5. Pivot element v_j: $\boldsymbol{A}_j$ enters the basis and $\boldsymbol{A}_{B(l)}$ exits.



(a) (b)

4.3 An example

Slide 9

$$
\begin{aligned}
\min \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \geq 2 \\
& x_1 \geq 1 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 = 2 \\
& x_1 - x_4 = 1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\max \quad & 2p_1 + p_2 \\
\text{s.t.} \quad & p_1 + p_2 \leq 1 \\
& 2p_1 \leq 1 \\
& p_1, p_2 \geq 0
\end{aligned}
$$

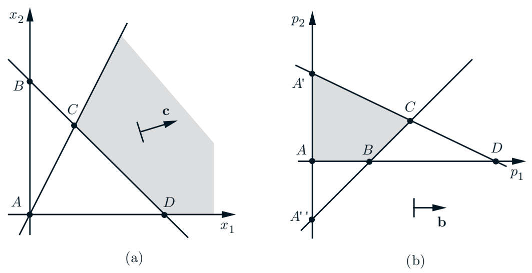
Slide 10

		x_1	x_2	x_3	x_4
	0	1	1	0	0
$x_3 =$	-2	-1	-2^*	1	0
$x_4 =$	-1	-1	0	0	1

Slide 11

		x_1	x_2	x_3	x_4
	-1	1/2	0	1/2	0
$x_2 =$	1	1/2	1	$-1/2$	0
$x_4 =$	-1	-1^*	0	0	1

		x_1	x_2	x_3	x_4
	$-3/2$	0	0	1/2	1/2
$x_2 =$	1/2	0	1	$-1/2$	1/2
$x_1 =$	1	1	0	0	-1



5 Duality and Degeneracy

Slide 12

- Any basis matrix $\boldsymbol{B}$ leads to dual basic solution $\boldsymbol{p}' = \boldsymbol{c_B}'\boldsymbol{B}^{-1}$.
- The dual constraint $\boldsymbol{p}'\boldsymbol{A}_j = c_j$ is active if and only if the reduced cost $\overline{c}_j$ is zero.
- Since $\boldsymbol{p}$ is m-dimensional, dual degeneracy implies more than m reduced costs that are zero.
- Dual degeneracy is obtained whenever there exists a nonbasic variable whose reduced cost is zero.

5.1 Example

Slide 13

$$\begin{array}{llcll}
\min & 3x_1 + x_2 & \qquad & \max & 2p_1 \\
\text{s.t.} & x_1 + x_2 - x_3 = 2 & & \text{s.t.} & p_1 + 2p_2 \le 3 \\
 & 2x_1 - x_2 - x_4 = 0 & & & p_1 - p_2 \le 1 \\
 & x_1, x_2, x_3, x_4 \ge 0, & & & p_1, p_2 \ge 0.
\end{array}$$

Equivalent primal problem

$$\begin{array}{lrcrcl}
\min & 3x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \ge & 2 \\
 & 2x_1 & - & x_2 & \ge & 0 \\
 & & & \multicolumn{3}{l}{x_1, x_2 \ge 0.}
\end{array}$$

Slide 14
Slide 15

- Four basic solutions in primal: A, B, C, D.
- Six distinct basic solutions in dual: A, A', A'', B, C, D.
- Different bases may lead to the same basic solution for the primal, but to different basic solutions for the dual. Some are feasible and some are infeasible.

5.2 Degeneracy and uniqueness

Slide 16

- If dual has a nondegenerate optimal solution, the primal problem has a unique optimal solution.
- It is possible, however, that dual has a degenerate solution and the dual has a unique optimal solution.

6.251J / 15.081J Introduction to Mathematical Programming
Fall 2009