Quiz 1

- Do not open this quiz booklet until you are directed to do so. Read all the instructions on this page.
- When the quiz begins, write your name on every page of this quiz booklet.
- This quiz contains 4 problems, some with multiple parts. You have 80 minutes to earn 80 points.
- This quiz booklet contains 13 pages, including this one. Two extra sheets of scratch paper are attached. Please detach them before turning in your quiz at the end of the examination period.
- This quiz is closed book. You may use one handwritten A4 or $8\frac{1}{2}'' \times 11''$ crib sheet. No calculators or programmable devices are permitted.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Do not put part of the answer to one problem on the back of the sheet for another problem, since the pages may be separated for grading.
- Do not waste time and paper rederiving facts that we have studied. It is sufficient to cite known results.
- Do not spend too much time on any one problem. Read them all through first, and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.
- Good luck!

Problem	Parts	Points	Grade	Grader
1	5	10		
2	1	7		
3	12	48		
4	2	15		
Total		80		

Name: ____________________

Problem 1. Asymptotic Running Times [10 points] (5 parts)

For each algorithm listed below,

- give a recurrence that describes its worst-case running time, and
- give its worst-case running time using Θ-notation.

You need not justify your answers.

(a) Binary search

(b) Insertion sort

(c) Randomized quicksort

(d) Strassen’s algorithm

(e) Merge sort

Problem 2. Substitution Method [7 points]

Consider the recurrence

$$T(n) = T(n/2) + T(n/4) + n\ .$$

Use the substitution method to give a tight upper bound on the solution to the recurrence using O-notation.

Problem 3. True or False, and Justify [48 points] (12 parts)

Circle **T** or **F** for each of the following statements to indicate whether the statement is true or false, respectively. If the statement is correct, briefly state why. If the statement is wrong, explain why. The more content you provide in your justification, the higher your grade, but be brief. Your justification is worth more points than your true-or-false designation.

T F Let f and g be asymptotically nonnegative functions. Then, at least one relationship of $f(n) = O(g(n))$ and $g(n) = O(f(n))$ must always hold.

T F The solution to the recurrence $T(n) = 3T(n/3) + O(\lg n)$ is $T(n) = \Theta(n \lg n)$.

T F Let F_k denote the kth Fibonacci number. Then, the n^2th Fibonacci number F_{n^2} can be computed in $O(\lg n)$ time.

T F Suppose that an array contains n numbers, each of which is -1, 0, or 1. Then, the array can be sorted in $O(n)$ time in the worst case.

T F An adversary can provide randomized quicksort with an input array of length n that forces the algorithm to run in $\omega(n \lg n)$ time on that input.

T F The array

$$20 \quad 15 \quad 18 \quad 7 \quad 9 \quad 5 \quad 12 \quad 3 \quad 6 \quad 2$$

forms a max-heap.

T F Heapsort can be used as the auxiliary sorting routine in radix sort, because it operates in place.

T F There exists a comparison sort of 5 numbers that uses at most 6 comparisons in the worst case.

T F Suppose that a hash table with collisions resolved by chaining contains n items and has a load factor of $\alpha = 1/\lg n$. Assuming simple uniform hashing, the expected time to search for an item in the table is $O(1/\lg n)$.

T F Let X be an indicator random variable such that $\mathrm{E}[X] = 1/2$. Then, we have $\mathrm{E}\left[\sqrt{X}\right] = 1/\sqrt{2}$.

T F Suppose that a hash table of m slots contains a single element with key k and the rest of the slots are empty. Suppose further that we search r times in the table for various other keys not equal to k. Assuming simple uniform hashing, the probability is r/m that one of the r searches probes the slot containing the single element stored in the table.

T F Let S be a set of n integers. One can create a data structure for S so that determining whether an integer x belongs to S can be performed in $O(1)$ time in the worst case.

Problem 4. Close Numbers [15 points] (2 parts)

Consider a set S of $n \geq 2$ distinct numbers. Call a pair of distinct numbers $x, y \in S$ ***close*** in S if

$$|x - y| \leq \frac{1}{n-1}\left(\max_{z \in S} z - \min_{z \in S} z\right) ,$$

that is, if the distance between x and y is at most the average distance between consecutive numbers in the sorted order.

(a) Explain briefly why every set S of $n \geq 2$ distinct numbers contains a pair of close elements.

(b) Suppose that we partition S around a pivot element $p \in S$, organizing the result into two subsets of S: $S_1 = \{x \in S \mid x \leq p\}$ and $S_2 = \{x \in S \mid x \geq p\}$. Prove that, for some $k \in \{1, 2\}$, there exists a pair $x, y \in S_k$ of numbers that are close in S (not just in S_k).

(c) Describe an $O(n)$-time algorithm to find a close pair of numbers in S. Analyze your algorithm. (*Hint:* Use divide and conquer.)

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