Mathematics for Computer Science
MIT 6.042J/18.062J

Uncountable Sets

Infinite Sizes

Are all sets the same size? NO!

Cantor's Theorem

shows how to keep finding bigger infinities.

Countable Sets

A is countable iff can list it:

$a_0, a_1, a_2, \ldots.$ example:

$\{0,1\}^* ::= \{$finite bit strings$\}$

Claim: $\{0,1\}^\omega ::= \{\infty$-bit strings$\}$ is uncountable.

Diagonal Arguments

Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^\omega$

	0	1	2	3	.	.	.	n	n+1	.	.	.
s_0	0	0	1	0	.	.	.	0	0	.	.	.
s_1	0	1	1	0	.	.	.	0	1	.	.	.
s_2	1	0	0	0	.	.	.	1	0	.	.	.
s_3	1	0	1	1	.	.	.	1	1	.	.	.
.	.	.			1							
.	.		.			1						
	.			.			0					

Diagonal Arguments

Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^\omega$

	0	1	2	3	.	.	.	n	n+1	.	.	.
s_0	1	0	1	0	.	.	.	0	0	.	.	.
s_1	0	0	1	0	.	.	.	0	1	.	.	.
s_2	1	0	1	0	.	.	.	1	0	.	.	.
s_3	1	0	1	0	.	.	.	1	1	.	.	.
.	.	.			0							
.	.		.			0						
	.			.			1					

Diagonal Arguments

Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^\omega$

…

…differs from every row!

So $\{0,1\}^\omega$ cannot be listed:

this diagonal sequence will be missing

$\{0,1\}^\omega$ **is uncountable**

So NOT$\left(\mathbb{N} \text{ surj } \{0,1\}^\omega\right)$ and

$\{0,1\}^\omega$ surj $\mathbb{N}$ obviously

$\mathbb{N}$ "strictly smaller" than $\{0,1\}^\omega$

Strictly Smaller

A strict B ::= NOT(A surj B)

A is "strictly smaller" than B

So $\mathbb{N}$ strict $\{0,1\}^\omega$

Cantor's Theorem

A strict pow(A)

for every set, A

(finite or infinite)

Diagonal Arguments

Suppose $A = \{a,b,s,t,\ldots,d,e,\ldots\}$

$pow(A) = \{f(a), f(b), f(s),\ldots, f(d),\ldots\}$

	a	b	s	t	.	.	.	d	e	.	.	.
f(a)												.
f(b)												.
f(s)												.
f(t)												.
.												

Diagonal Arguments

Suppose $A = \{a,b,s,t,\ldots,d,e,\ldots\}$

$pow(A) = \{f(a), f(b), f(s),\ldots, f(d),\ldots\}$

	a	b	s	t	c	.	.	d	e	.	.	.
f(a)	a		s	t					e			.
f(b)	a	b			c			d				.
f(s)		b		t								.
f(t)			s	t	c			d				.
f(c)		b	s					d	e			
.												
.												

Diagonal Arguments

Suppose $A = \{a,b,s,t,\ldots,d,e,\ldots\}$

$pow(A) = \{f(a), f(b), f(s),\ldots, f(d),\ldots\}$

	a	b	s	t	c	.	.	d	e	.	.	.
f(a)	~~a~~		s	t					e			.
f(b)	a	~~b~~			c			d				.
f(s)		b	s	t								.
f(t)			s	~~t~~	c			d				.
f(c)		b	s		c			d	e			
.						.						
.							.					

Diagonal Arguments

Suppose $A = \{a,b,s,t,\ldots,d,e,\ldots\}$

$pow(A) = \{f(a), f(b), f(s),\ldots, f(d),\ldots\}$

D	a	b	s	t	c	.	.	d	e	.	.	.
f(a)			s	t					e			.
f(b)	a				c			d				.
f(s)		b	s	t								.
f(t)			s		c			d				.
f(c)		b	s		c			d	e			
.						.						
.							.					

A strict Pow(A)

Pf: say have fcn $f:A \to pow(A)$.
Define a subset of A that is not in the range of f: namely

$$D ::= \{a \in A \mid a \notin f(a)\}$$

Now $D \notin range(f)$ since it differs from set $f(a)$ at element a!

A strict Pow(A)

So no f-arrow into D.
f is not a surjection.
QED

$\mathbb{N}$ strict pow($\mathbb{N}$)

That is,
$pow(\mathbb{N})$ is uncountable

Proving Uncountability

Lemma: If A is uncountable and C surj A then C is uncountable

$\{0,1\}^\omega$ again

We know

$$\{0,1\}^\omega \text{ bij } \text{pow}(\mathbb{N})$$

and $\text{pow}(\mathbb{N})$ uncountable by Cantor,

$\{0,1\}^\omega$ again

We know

$$\{0,1\}^\omega \text{ bij } \text{pow}(\mathbb{N})$$

and $\text{pow}(\mathbb{N})$ uncountable by Cantor, so $\{0,1\}^\omega$ uncountable.

Real Numbers Uncountable

$$\mathbb{R} \text{ surj } \{0,1\}^\omega$$

map $\pm r$ to binary rep

$3\ 1/3 = 111.010101\ldots$

maps to $111010101\ldots$

Real Numbers Uncountable

$$\mathbb{R} \text{ surj } \{0,1\}^{\omega}$$

map $\pm r$ to binary rep

$1/2 \;=\; .100000\ldots$

$1/2$ maps to $100000\ldots$

$\;=\; .0111111\ldots$

$-1/2$ maps to $0111111\ldots$