

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science

MIT 6.042J/18.062J

# Uncountable Sets



Albert R Meyer, March 4, 2015

Cantor.1

6	9	13	7
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## Infinite Sizes

Are all sets the same size? **NO!**

Cantor's Theorem

shows how to keep finding bigger infinities.



Albert R Meyer, March 4, 2015

Cantor.2

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## Countable Sets

A is countable iff can list it:

$a_0, a_1, a_2, \dots$  example:

$\{0,1\}^*$  ::= {finite bit strings}

Claim:  $\{0,1\}^\omega$  ::= { $\infty$ -bit strings} is uncountable.



Albert R Meyer, March 4, 2015

Cantor.3

6	9	13	7
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## Diagonal Arguments

Suppose  $s_0, s_1, s_2, \dots \in \{0,1\}^\omega$

	0	1	2	3	...	n	n+1	...
$s_0$	0	0	1	0	...	0	0	...
$s_1$	0	1	1	0	...	0	1	...
$s_2$	1	0	0	0	...	1	0	...
$s_3$	1	0	1	1	...	1	1	...
	.	.	.	1				
	.	.	.	1				
	.	.	.	0				



Albert R Meyer, March 4, 2013

Cantor.4

6	9	13	7
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## Diagonal Arguments

Suppose  $s_0, s_1, s_2, \dots \in \{0,1\}^\omega$

	0	1	2	3	...	n	n+1	...
$s_0$	1	0	1	0	...	0	0	...
$s_1$	0	0	1	0	...	0	1	...
$s_2$	1	0	1	0	...	1	0	...
$s_3$	1	0	1	0	...	1	1	...
				0				
					0			
						1		



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Cantor.5

6	9	13	7
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## Diagonal Arguments

Suppose  $s_0, s_1, s_2, \dots \in \{0,1\}^\omega$

...  
 ...differs from every row!  
 So  $\{0,1\}^\omega$  cannot be listed:  
 this diagonal sequence  
 will be missing



Albert R Meyer, March 4, 2013

Cantor.6

6	9	13	7
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$\{0,1\}^\omega$  is uncountable  
 So NOT  $\mathbb{N} \text{ surj } \{0,1\}^\omega$  and  
 $\{0,1\}^\omega \text{ surj } \mathbb{N}$  obviously  
 $\mathbb{N}$  "strictly smaller" than  $\{0,1\}^\omega$



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Cantor.8

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## Strictly Smaller

A strict B ::= NOT(A surj B)  
 A is "strictly smaller" than B

So  $\mathbb{N}$  strict  $\{0,1\}^\omega$



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Cantor.9

6	9	13	7
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## Cantor's Theorem

A strict  $\text{pow}(A)$   
for every set,  $A$   
(finite or infinite)



Albert R Meyer, March 4, 2015

Cantor.10

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## Diagonal Arguments

Suppose  $A = \{a, b, s, t, \dots, d, e, \dots\}$   
 $\text{pow}(A) = \{f(a), f(b), f(s), \dots, f(d), \dots\}$

	a	b	s	t	.	.	d	e	.	.
f(a)										.
f(b)										.
f(s)										.
f(t)										.
.										.
.										.



Albert R Meyer, March 4, 2013

Cantor.11

6	9	13	7
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## Diagonal Arguments

Suppose  $A = \{a, b, s, t, \dots, d, e, \dots\}$   
 $\text{pow}(A) = \{f(a), f(b), f(s), \dots, f(d), \dots\}$

	a	b	s	t	c	.	.	d	e	.	.
f(a)	a		s	t				e		.	.
f(b)	a	b			c			d		.	.
f(s)		b		t						.	.
f(t)			s	t	c			d		.	.
f(c)		b	s					d	e	.	.
.										.	.
.										.	.



Albert R Meyer, March 4, 2013

Cantor.12

6	9	13	7
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## Diagonal Arguments

Suppose  $A = \{a, b, s, t, \dots, d, e, \dots\}$   
 $\text{pow}(A) = \{f(a), f(b), f(s), \dots, f(d), \dots\}$

	a	b	s	t	c	.	.	d	e	.	.
f(a)	<del>a</del>		s	t				e		.	.
f(b)	a	<del>b</del>			c			d		.	.
f(s)		b	<del>s</del>	t						.	.
f(t)			s	<del>t</del>	c			d		.	.
f(c)		b	s		<del>c</del>			d	e	.	.
.										.	.
.										.	.



Albert R Meyer, March 4, 2013

Cantor.13

6	9	13	7
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## Diagonal Arguments

Suppose  $A = \{a, b, s, t, \dots, d, e, \dots\}$   
 $\text{pow}(A) = \{f(a), f(b), f(s), \dots, f(d), \dots\}$

	a	b	s	t	c	.	d	e	.
D									
f(a)			s	t				e	.
f(b)	a				c			d	.
f(s)		b	s	t					.
f(t)			s		c			d	.
f(c)		b	s		c			d	e
.									.
.									.



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Cantor.14

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## A strict Pow(A)

Pf: say we have fcn  $f: A \rightarrow \text{pow}(A)$ .  
 Define a subset of  $A$  that is not in  
 the range of  $f$ : namely

$$D ::= \{a \in A \mid a \notin f(a)\}$$

Now  $D \notin \text{range}(f)$  since it differs  
 from set  $f(a)$  at element  $a$ !



Albert R Meyer, March 4, 2015

Cantor.15

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## A strict Pow(A)

So no  $f$ -arrow into  $D$ .  
 $f$  is not a surjection.  
**QED**



Albert R Meyer, March 4, 2015

Cantor.21

6	9	13	7
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## $\mathbb{N}$ strict pow( $\mathbb{N}$ )

That is,  
 $\text{pow}(\mathbb{N})$  is uncountable



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Cantor.22

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## Proving **U**ncountability

**L**emma: If  $A$  is **u**ncountable  
and  $C \text{ surj } A$  then  
 $C$  is **u**ncountable



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Cantor.24

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## $\{0,1\}^\omega$ again

We know

$\{0,1\}^\omega$  bij  $\text{pow}(\mathbb{N})$   
and  $\text{pow}(\mathbb{N})$  uncountable by  
Cantor,



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Cantor.26

6	9	13	7
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## $\{0,1\}^\omega$ again

We know

$\{0,1\}^\omega$  bij  $\text{pow}(\mathbb{N})$   
and  $\text{pow}(\mathbb{N})$  uncountable by  
Cantor, so  $\{0,1\}^\omega$  uncountable.



Albert R Meyer, March 4, 2015

Cantor.27

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## Real Numbers Uncountable

$\mathbb{R} \text{ surj } \{0,1\}^\omega$

map  $\pm r$  to binary rep  
 $3 \frac{1}{3} = 111.010101\dots$   
maps to  $111010101\dots$



Albert R Meyer, March 4, 2015

Cantor.28

6	9	13	7
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## Real Numbers Uncountable

$$\mathbb{R} \text{ surj } \{0,1\}^{\omega}$$

map  $\pm r$  to binary rep

$$1/2 = .100000\dots$$

$1/2$  maps to  $100000\dots$

$$= .011111\dots$$

$-1/2$  maps to  $011111\dots$



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