Mathematics for Computer Science
MIT 6.042J/18.062J

Asymptotic Properties

The Oh's

lemma:

If $f = o(g)$ or $f \sim g$, then $f = O(g)$

$\lim = 0$ OR $\lim = 1$ IMPLIES $\lim < \infty$

The Oh's

If $f = o(g)$, then $g \neq O(f)$

$\lim \frac{f}{g} = 0$ IMPLIES $\lim \frac{g}{f} = \infty$

Little Oh: $o(\cdot)$

Lemma: $x^a = o(x^b)$ for $a < b$

Proof: $\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$ and $b - a > 0$

so as $x \to \infty$, $\frac{1}{x^{b-a}} \to 0$

Little Oh: $o(\cdot)$

Lemma:

$$\ln x = o(x^{\varepsilon})$$

for $\varepsilon > 0$.

Little Oh: $o(\cdot)$

Lemma: $\ln x = o(x^{\varepsilon})$ for $\varepsilon > 0$.

Proof:

$\frac{1}{y} \leq y$ for $y \geq 1$ so $\int_1^z \frac{1}{y}\,dy \leq \int_1^z y\,dy$

$$\ln z \leq \frac{z^2}{2} \text{ for } z \geq 1$$

Little Oh: $o(\cdot)$

Lemma: $\ln x = o(x^{\varepsilon})$ for $\varepsilon > 0$.

Proof: $\ln z \leq \frac{z^2}{2}$, so let $z = \sqrt{x^{\delta}}$

$$\frac{\delta \ln x}{2} \leq \frac{x^{\delta}}{2}$$

$$\ln x \leq \frac{x^{\delta}}{\delta} = o(x^{\varepsilon}) \text{ for } \varepsilon > \delta.$$

Little Oh: $o(\cdot)$

Lemma:

$$x^c = o(a^x)$$

for $a > 1$.

Little Oh: $o(\cdot)$

proofs:
L'Hopital's Rule,
McLaurin Series
(see a Calculus text)

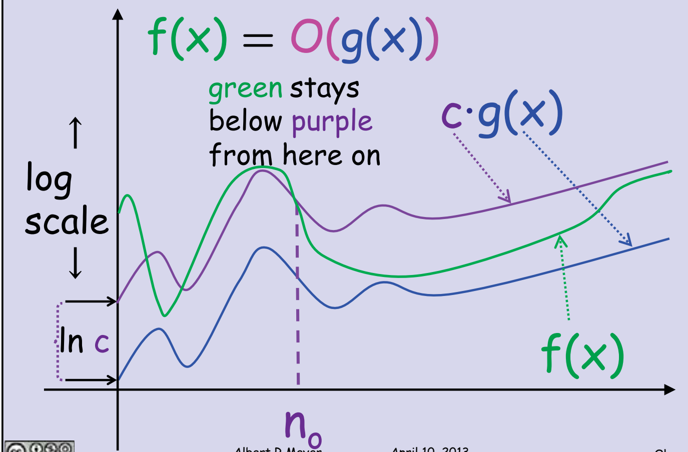
Big Oh: $O(\cdot)$

Equivalent definition:

$$f(n) = O(g(n))$$

$$\exists c, n_0\ \forall n \geq n_0.\ f(n) \leq c \cdot g(n)$$

Big Oh: $O(\cdot)$



Why limsup?

If $f \leq 2g$ then $f = O(g)$,
but maybe f/g has no limit.

example $f(n) = \left(1 + \sin^2\left(\frac{n\pi}{2}\right)\right) \cdot g(n)$

but

$$\limsup_{n \to \infty} \frac{f(n)}{g(n)} = 2 < \infty$$

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