

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Mathematics for Computer Science
MIT 6.042J/18.062J

Asymptotic Notation



Albert R Meyer,

April 10, 2013

theOhs.1

6	9	13	7
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Asymptotic Equivalence

Def: $f(n) \sim g(n)$

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$$



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6	9	13	7
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Asymptotic Equivalence \sim

$$n^2 \sim n^2 + n$$

because

$$\lim_{n \rightarrow \infty} \frac{n^2 + n}{n^2} = \lim_{n \rightarrow \infty} 1 + \frac{1}{n} = 1$$



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6	9	13	7
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Asymptotic Equivalence \sim

Lemma: \sim is symmetric

Proof: Say $f \sim g$. Now

$$\lim \frac{g}{f} = \lim \frac{1}{\left(\frac{f}{g}\right)} = \frac{1}{\lim \left(\frac{f}{g}\right)} = \frac{1}{1}$$



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6	9	13	7
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Asymptotic Equivalence \sim

Lemma: \sim is symmetric

Proof: so $g \sim f$. ■

$$\lim \frac{g}{f} = \lim \frac{1}{\left(\frac{f}{g}\right)} = \frac{1}{\lim \left(\frac{f}{g}\right)} = \frac{1}{1}$$



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6	9	13	7
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transitivity of \sim

Suppose $f \sim g$ and $g \sim h$,
prove $f \sim h$.

$$1 = \lim \frac{f}{g} = \lim \frac{\left(\frac{f}{h}\right)}{\left(\frac{g}{h}\right)} = \frac{\lim \left(\frac{f}{h}\right)}{\lim \left(\frac{g}{h}\right)}$$



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6	9	13	7
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transitivity of \sim

Suppose $f \sim g$ and $g \sim h$,
prove $f \sim h$.

$$1 = \lim \frac{f}{g} = \lim \frac{\left(\frac{f}{h}\right)}{\left(\frac{g}{h}\right)} = \frac{\lim \left(\frac{f}{h}\right)}{1}$$



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6	9	13	7
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Asymptotic Equivalence \sim

Corollary: \sim is an
equivalence relation



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6	9	13	7
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Asymptotic Equivalence \sim

\sim is a relation
on functions:

$$f \sim g$$



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theOhs.10

6	9	13	7
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Little Oh: $o(\cdot)$

Asymptotically smaller

Def: $f(n) = o(g(n))$

iff

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$$



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6	9	13	7
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Little Oh: $o(\cdot)$

$$n^2 = o(n^3)$$

because

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^3} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$



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6	9	13	7
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Little Oh: $o(\cdot)$

Lemma:

$o(\cdot)$ is a strict
partial order



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Big Oh: $O(\cdot)$

Asymptotic Order of Growth:

$$f = O(g)$$

$$\limsup_{n \rightarrow \infty} \left(\frac{f(n)}{g(n)} \right) < \infty$$

a technicality — ignore now



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Big Oh: $O(\cdot)$

$$3n^2 = O(n^2)$$

because

$$\lim_{n \rightarrow \infty} \frac{3n^2}{n^2} = 3 < \infty$$



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Theta: $\Theta(\cdot)$

Same Order of Growth:

$$f = \Theta(g)$$

$$\text{Def: } f = O(g)$$

and

$$g = O(f)$$



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6	9	13	7
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Theta: $\Theta(\cdot)$

Lemma:

$\Theta(\cdot)$ is an
equivalence
relation



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Asymptotics: Intuitive Summary

- $f \sim g$: f & g nearly equal
- $f = o(g)$: f much less than g
- $f = O(g)$: f roughly $\leq g$
- $f = \Theta(g)$: f roughly equal g



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