In-Class Problems Week 1, Fri.

Problem 1.
Prove that if $a \cdot b = n$, then either a or b must be $\leq \sqrt{n}$, where a, b, and n are nonnegative real numbers.
Hint: by contradiction, Section 1.8 in the course textbook.

Problem 2.
Generalize the proof of Theorem 1.8.1 repeated below that $\sqrt{2}$ is irrational[1] in the course textbook.
For example, how about $\sqrt{3}$?

Theorem. *$\sqrt{2}$ is an irrational number.*

Proof. The proof is by contradiction: assume that $\sqrt{2}$ is rational, that is,

$$\sqrt{2} = \frac{n}{d}, \tag{1}$$

where n and d are integers. Now consider the smallest such positive integer denominator, d. We will prove in a moment that the numerator, n, and the denominator, d, are both even. This implies that

$$\frac{n/2}{d/2}$$

is a fraction equal to $\sqrt{2}$ with a smaller positive integer denominator, a contradiction.

> *Since the assumption that $\sqrt{2}$ is rational leads to this contradiction, the assumption must be false. That is, $\sqrt{2}$ is indeed irrational.* This italicized comment on the implication of the contradiction normally goes without saying, but since this is an early example of proof by contradiction, we've said it.

To prove that n and d have 2 as a common factor, we start by squaring both sides of (1) and get $2 = n^2/d^2$, so

$$2d^2 = n^2. \tag{2}$$

So 2 is a factor of n^2, which is only possible if 2 is in fact a factor of n.

This means that $n = 2k$ for some integer, k, so

$$n^2 = (2k)^2 = 4k^2. \tag{3}$$

Combining (2) and (3) gives $2d^2 = 4k^2$, so

$$d^2 = 2k^2. \tag{4}$$

So 2 is a factor of d^2, which again is only possible if 2 is in fact also a factor of d, as claimed. ∎

[1] Remember that an irrational number is a number that cannot be expressed as a ratio of two integers.

Problem 3.
If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering $\sqrt{2}^{\sqrt{2}}$ and arguing by cases.

Problem 4.
The fact that that there are irrational numbers a, b such that a^b is rational was proved earlier by cases. Unfortunately, that proof was *nonconstructive*: it didn't reveal a specific pair, a, b, with this property. But in fact, it's easy to do this: let $a ::= \sqrt{2}$ and $b ::= 2\log_2 3$.

We know $a = \sqrt{2}$ is irrational, and $a^b = 3$ by definition. Finish the proof that these values for a, b work, by showing that $2\log_2 3$ is irrational.