So, now we'll look at third kind of sum that comes up all the time called harmonic sums. And we'll begin by examining an example where they come up really directly. So, here's the puzzle.

Suppose that I'm trying to stack a bunch of books on a table. Assume all the books are the same size and weight and uniform, and I'd like to stack them up one on top of the other in some way. And try to get them as far out past the end of the table as I can manage. Now, notice in this picture, it seems kind of paradoxical. The top book, the back end of the top book is past the edge of the table.

Is it possible to do that? Is it possible to get the top book, the back of the top book past the edge of the table? And how far out can you get the further most book to the right? That's the question we want to ask.

Well, let's go back and do it for the simplest case, which is one book. So, this amount will be a function of how many books we have. We're interested in the overhang using n books. Overhang is the amount past the edge of the table that the rightmost end of any book can be.

What do you do with one book? Well, with one book, assuming that the thing is uniform, the center of mass in the middle. Let's assume the book is of length 1. So, the center of mass of the book is at halfway down the book. And if that center of mass is not over the table, then you're going to have torque and the book is going to fall.

So, you've got to keep the center of mass supported. And the way to get the largest overhang is to have the center of mass right at the edge of the table here. And in that case, you can get the book to stick out half a book length without falling. And what that tells us is that the one book overhang is 1/2. It'll balance with the furthest end out exactly if the center of mass is on the edge, and I get a half a book length for unit overhang.

Let's proceed recursively or inductively. Suppose I have n books. How am I going to get them to balance? Well, let's assume that I figured out how to get a so-called stable stack of n books, which if I completely supported it flat on the table, it wouldn't fall over. And I'm going to show you how to go from n to n plus 1, which is how you construct an arbitrarily large stack of books that won't fall over.

Well, if the stack completely resting on the table won't fall over, that means that if I have the center of mass of it past the edge of the table, by definition of the center of mass, there's going to be an equal amount of weight on both sides of the center of mass, and the thing is going to fall off the edge of the table by the same reasoning as we did for one book. So, the stable and n stack-- stable in the sense that it won't fall over of self if it was lying completely over the table. In fact, it won't fall over as long as its center of mass is over the table. And to get it out the furthest amount to the right, what I'm going to do is put it at the edge of the table.

So, now I know how to place a stable stack of n books to get the largest overhang out of it. What next? Well, let's consider n plus 1 books. And what do I have to do? So, I'm trying to do the same deal. Suppose that I have a nice stack of n books and I know how to support it so it won't tip over. And I now have n plus 1 books and I want to get out further. What do I have to do?

Well, by the basic reasoning that we just went through, now the center of mass of the whole stack of n plus 1 books has to be over the edge of the table. That's the way I'm going to get out furthest. So, I know where the center of mass of n plus books is going to be, at the edge of the table.

What about the center of mass of the top n books? Well, I need them to be supported. I need their center of mass to be supported. They'll be supported, providing their center of mass is over the bottom book somewhere. And the way to get it out furthest is to have it over the right edge of the bottom book.

So, I'm going to put the center of mass of the top n books at the edge of the n plus first book here. And that means that the incremental overhang that I get, the increase in overhang that I get by adding one more book, we can call the delta overhang. And it's the distance between the center of mass of n plus 1 books and the center of mass of n books. N here, and n plus 1 here.

Well, let's see what's going on. The center of mass of the n books is at some location here. And the center of mass of the bottom book is halfway away, half a book length away from where the n books are balanced on the edge of the bottom book. So, the center of mass of the n books is here. The center of mass of the bottom book is there. The distance between them is 1/2.

And I need the table to be at the balance point between the n books and the one book. That's where the center of mass of the n plus 1 books will be. So, I need to calculate amount that's going to be, the increase in overhang. So, let's abstract it a little bit.

The delta overhang is the distance from the n book to the n plus 1 book centers of mass. And if we think of this as a balancing diagram, there's the n books. Or at least, there's the center of mass of the n books. There's the center of mass of the 1 book, their distance 1/2 apart, which we said. And they have to balance at the edge of the table.

So, think of the edge of the table as the pivot point and it's there. And I need to calculate, where is that pivot point? How do I get this fulcrum or this balance beam to balance with weight n here and weight 1 there, when their total length apart is 1/2. What's this distance? That distance is the delta that I'm trying to calculate.

Well, you just know from physics that the balance point is going to be the distance 1/2 divided by the sum of n and n plus 1. I need the n times this amount to equal 1 times that amount. And if you check that out, it means that delta is 1/2 over n plus 1. Or simplifying, 1 over twice n plus 1. You should stare at that diagram a little bit and remember your elementary physics to realize the reasoning behind the formula for delta.

Well, now I'm done because basically, I've just figured out that the increase is this delta overhang. And now I know what it is. It's 1 over twice n plus 1. And so, what I can conclude is that the overhang of n books, B1 is 1/2 and Bn plus 1 is Bn plus 1 over twice n plus 1. So this is a recursive definition of Bn, but it's easy to see how it unwinds. It means that Bn is 1/2 plus 1/2 of 1 plus 1 plus 1/2 of 2 plus 1 plus 1/2 of 3 plus 1 and so on. If I factor out the 1/2, Bn is 1/2 times 1 plus 1/2 plus 1/3 out through 1 over n.

That sum is the harmonic sum. The sum 1 plus 1/2 up through 1 over n is called Hn, or the harmonic sum. And what we figured out, or really the harmonic number, is the value of that sum. And what we figured out is that Bn, the amount that I can get n books out up past the edge of the table is Hn over 2.