Problem Set 3

Due: February 27

Reading:

- Section 4.3. *Functions* through 4.5. *Finite Cardinality* in the course textbook.
- Chapter 5. *Induction* through 5.3. *Induction vs WOP* in the course textbook.

Problem 1.
The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined as follows:

$$F_n ::= \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ F_{n-1} + F_{n-2} & \text{if } n > 1. \end{cases}$$

Prove, using strong induction, the following closed-form formula for F_n.[1]

$$F_n = \frac{p^n - q^n}{\sqrt{5}}$$

where $p = \frac{1+\sqrt{5}}{2}$ and $q = \frac{1-\sqrt{5}}{2}$.

Hint: Note that p and q are the roots of $x^2 - x - 1 = 0$, and so $p^2 = p + 1$ and $q^2 = q + 1$.

Problem 2.
The Block Stacking Game[2] goes as follows: You begin with a stack of n boxes. Then you make a sequence of moves. In each move, you divide one stack of boxes into two nonempty stacks. The game ends when you have n stacks, each containing a single box. You earn points for each move; in particular, if you divide one stack of height $a + b$ into two stacks with heights a and b, then you score ab points for that move. Your overall score is the sum of the points that you earn for each move. What strategy should you use to maximize your total score?

As an example, suppose that we begin with a stack of $n = 10$ boxes. Then the game might proceed as shown in Figure 1.

Define the *potential*, $p(S)$, of a stack of blocks, S, to be $k(k-1)/2$ where k is the number of blocks in S. Define the potential, $p(A)$, of a set of stacks, A, to be the sum of the potentials of the stacks in A.

Show that for any set of stacks, A, if a sequence of moves starting with A leads to another set of stacks, B, then $p(A) \geq p(B)$, and the score for this sequence of moves is $p(A) - p(B)$.

Hint: Try induction on the number of moves to get from A to B.

[1]This mind-boggling formula is known as *Binet’s formula*. We’ll explain in Chapter 15, and again in Chapter 21, how it comes about.

[2]Excerpted from [28] Section 5.2.4 in the course textbook.

Stack Heights										**Score**
$\underline{10}$										
5	$\underline{5}$									25 points
$\underline{5}$	3	2								6
$\underline{4}$	3	2	1							4
2	$\underline{3}$	2	1	2						4
$\underline{2}$	2	2	1	2	1					2
1	$\underline{2}$	2	1	2	1	1				1
1	1	$\underline{2}$	1	2	1	1	1			1
1	1	1	1	$\underline{2}$	1	1	1	1		1
1	1	1	1	1	1	1	1	1	1	1
							Total Score	=		45 points

Figure 1 An example of the stacking game with $n = 10$ boxes. On each line, the underlined stack is divided in the next step.

Problem 3.
Let A, B, and C be sets, and let $f : B \to C$ and $g : A \to B$ be functions. Let $h : A \to C$ be the composition, $f \circ g$, that is, $h(x) ::= f(g(x))$ for $x \in A$. Prove or disprove the following claims:

Hint: Arguments based on "arrows" using Definition 4.4.2 are fine.

(a) If h is surjective, then f must be surjective.

(b) If h is surjective, then g must be surjective.

(c) If h is injective, then f must be injective.

(d) If h is injective and f is total, then g must be injective.