Power & Limits of Logic

Three Profound Theorems
about
Mathematical Logic

Gödel's Completeness Theorem

Thm 1, good news:
only need to know* a few axioms & rules, to prove *all* validities.

**Theoretically only: having more rules is convenient.*

Axioms & Inference Rules

Namely, starting from a few propositional & simple predicate validities, every valid assertion can be proved using just UG and *modus ponens* repeatedly!

Cannot Determine Validity

Thm 2, Bad News: there is *no procedure* that determines when quantified assertions are valid (in contrast to propositional formulas).

Gödel's *In*completeness Theorem for Arithmetic

Thm 3, Worse News:
if we stick to domain, $\mathbb{N}$, with predicates $x + y = z$,
$x \cdot y = z$,

then no proof system can prove all the true assertions!

Three Profound Theorems

We won't prove these Theorems. Their proofs usually require half a term in an intro logic course after 6.042.

Mathematics for Computer Science
MIT 6.042J/18.062J

Sets & Functions

Sets

Informally,

a set is a collection of mathematical objects with the collection treated as a single mathematical object.

This is circular of course: what's a *collection*?

Some sets

Real numbers, $\mathbb{R}$

complex numbers, $\mathbb{C}$□

integers, $\mathbb{Z}$

the emptyset, $\varnothing$

the $\underbrace{\text{set of all sets}}_{\text{the power set}}$ of integers, pow($\mathbb{Z}$)

Some sets

{7, `“Albert R.”`, π/2, T}

A set with 4 elements: two numbers, a string, and a Boolean value.
Same as

{`“Albert R.”`,7,T, π/2}

-- order doesn't matter

Membership

$x \in A$ x is an element of A

π/2 ∈ {7, `“Albert R.”`, π/2, T}

π/3 ∉ {7, `“Albert R.”`, π/2, T}

14/2 ∈ {7, `“Albert R.”`, π/2, T}

Membership

$x \in A$ x is a member of A

shorter: "x is in A"

$7 \in \mathbb{Z}$ $2/3 \notin \mathbb{Z}$ $\mathbb{Z} \in$ pow($\mathbb{R}$)

In or Not In

{7, π/2, 7} is same as {7, π/2}

An element, like 7, is in a set,

or not in the set.

(No notion of being in the set

more than once)

Containment

$A \subseteq B$ *A* is a subset of *B*

A is contained in *B*

Every element of *A* is also an element of *B*.

$\mathbb{Z} \subseteq \mathbb{R}$, $\mathbb{R} \subseteq \mathbb{C}$, $\{3\} \subseteq \{5,7,3\}$

$\varnothing \subseteq$ every set

Team Problem

Problem 1

Defining Sets

The set of elements, *x*, in *A*

such that *P*(*x*) is true.

$$\{x \in A \mid P(x)\}$$

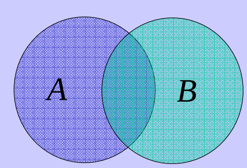
Defining Sets

The set of even integers:

$$\{n \in \mathbb{Z} \mid n \text{ is even}\}$$

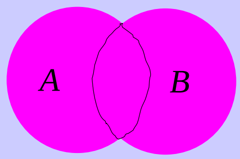
$$\{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}.\ n = 2m\}$$

New sets from old



Venn Diagram for *A* and *B*

union



$$A \cup B ::= \{x \mid (x \in A) \vee (x \in B)\}$$

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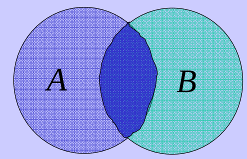
intersection



$$A \cap B ::= \{x \mid x \in A \wedge x \in B\}$$

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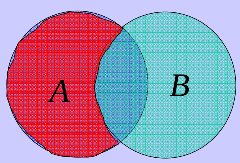
set difference



$$A - B ::= \{x \mid (x \in A) \wedge (x \notin B)\}$$

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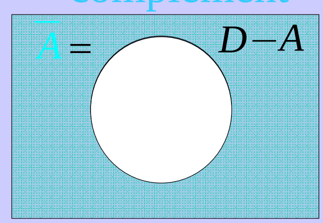
complement



known domain D

$$\overline{A} ::= \{x \in D \mid x \notin A\} = D - A$$

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power set

$$\text{pow}(A) ::= \{S \mid S \subseteq A\}$$

$$\text{pow}(\{a,b\}) = \{\{a,b\},\{a\},\{b\},\varnothing\}$$

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Quickie

What is Pow(∅)?

Ans: {∅}

What is Pow(Pow(∅))?

Ans: {{∅}, ∅}

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Russell's Paradox

Let $W ::= \{S \in \text{Sets} \mid S \notin S\}$

so $S \in W \leftrightarrow S \notin S$

Let S be W and reach a contradiction:

$$W \in W \leftrightarrow W \notin W$$

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Russell's Paradox

So the collection, Sets, of all sets, cannot be considered a set!

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Functions

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$f : A \to B$

function, f, from set A to set B associates an element, $f(a) \in B$ with an element $a \in A$.

Example: f is the string-length function: f(“aabd”)=4

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$f : A \to B$

f is the string-length function.

A, the domain of f, is the set of strings.

B, the codomain of f, is $\mathbb{N}$

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$g : \mathbb{R}^2 \to \mathbb{R}$

$$g(x, y) = \frac{1}{x - y}$$

domain(g) = all pairs of reals

codomain(g)= all reals

But g is partial: not defined on all pairs of reals

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$$g' : D \to \mathbb{R}$$

$$g'(x, y) = \frac{1}{x - y}$$

where $D = \mathbb{R}^2 - \{(x, y) \mid y = x\}$

Then g' is total:

defined on all pairs in domain D.

Total functions

$f : A \to B$ is total

iff every element of A is assigned a B-value by f

$$\forall a \in A\, \exists b \in B.\ f(a) = b$$

Onto functions

$f : A \to B$ is onto

iff every element of B is f of something

$$\forall b \in B\, \exists a \in A.\ f(a) = b$$

a surjection

1-1 functions

$f : A \to B$ is 1-1

iff every element of B is f of *at most* 1 thing

$$\forall a, a' \in A.\ (f(a) = f(a')) \to (a = a')$$

an injection

Bijections

$f : A \to B$ is a bijection iff

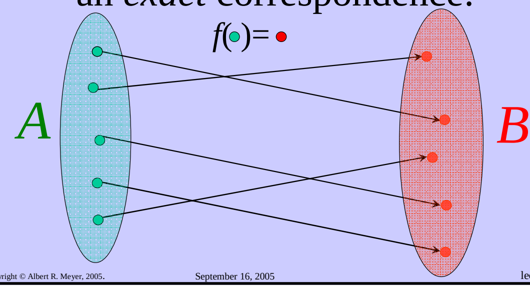
it is all those good things:

total, onto, and 1-1

Bijections

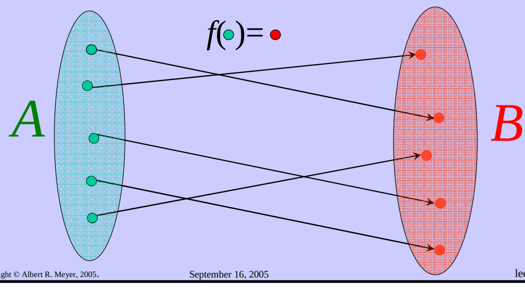
$f : A \to B$ is a bijection iff it is

an *exact* correspondence:



Bijections

exactly one arrow out exactly one arrow in



Team Problem

Problem 2