In-Class Problems Week 8, Wed.

Problem 1. We begin with two large glasses. The first glass contains a pint of water, and the second contains a pint of wine. We pour 1/3 of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour 1/3 of a pint of the mix back into the first glass and repeat this pouring back-and-forth process a total of n times.

(a) Describe a closed form formula for the amount of wine in the first glass after n back-and-forth pourings.

(b) What is the limit of the amount of wine in each glass as n approaches infinity?

Problem 2. Suppose you were about to enter college today and a college loan officer offered you the following deal: $25,000 at the start of each year for four years to pay for your college tuition and an option of choosing one of the following repayment plans:

Plan A: Wait four years, then repay $20,000 at the start of each year for the next ten years.

Plan B: Wait five years, then repay $30,000 at the start of each year for the next five years.

Suppose the annual interest rate paid by banks is 7% and does not change in the future.

(a) Assuming that it's no hardship for you to meet the terms of either payback plan, which one is a better deal? (You will need a calculator.)

(b) What is the loan officer's effective profit (in today's dollars) on the loan?

Problem 3. Riemann's Zeta Function $\zeta(k)$ is defined to be the infinite summation:

$$1 + \frac{1}{2^k} + \frac{1}{3^k} \cdots = \sum_{j \geq 1} \frac{1}{j^k}$$

Below is a proof that

$$\sum_{k \geq 2} (\zeta(k) - 1) = 1$$

Justify each line of the proof. (P.S. The purpose of this exercise is to highlight some of the rules for manipulating series. Don't worry about the significance of this identity.)

$$\begin{aligned}
\sum_{k \geq 2} (\zeta(k) - 1) &= \sum_{k \geq 2} \left[\left(\sum_{j \geq 1} \frac{1}{j^k} \right) - 1 \right] &(1)\\
&= \sum_{k \geq 2} \sum_{j \geq 2} \frac{1}{j^k} &(2)\\
&= \sum_{j \geq 2} \sum_{k \geq 2} \frac{1}{j^k} &(3)\\
&= \sum_{j \geq 2} \frac{1}{j^2} \sum_{k \geq 0} \frac{1}{j^k} &(4)\\
&= \sum_{j \geq 2} \frac{1}{j^2} \cdot \frac{1}{1 - 1/j} &(5)\\
&= \sum_{j \geq 2} \frac{1}{j(j-1)} &(6)\\
&= \lim_{n \to \infty} \sum_{j=2}^{n} \frac{1}{j(j-1)} &(7)\\
&= \lim_{n \to \infty} \sum_{j=2}^{n} \frac{1}{j-1} - \frac{1}{j} &(8)\\
&= \lim_{n \to \infty} (1 - \frac{1}{n}) &(9)\\
&= 1 &(10)
\end{aligned}$$