Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

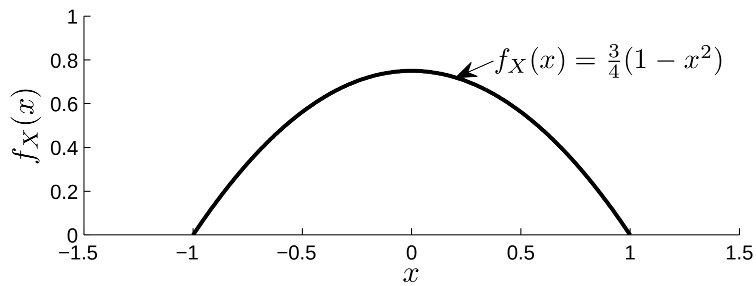
Problem Set 6
Due October 27, 2010

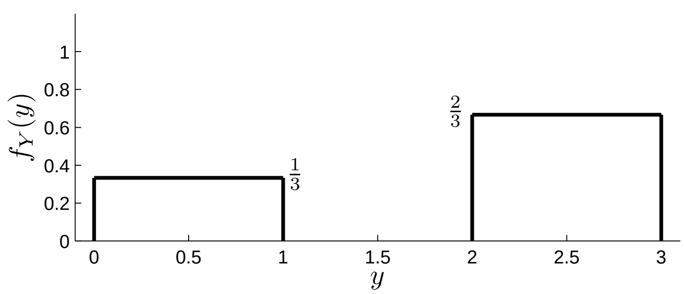
1. Random variables X and Y are distributed according to the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} ax, & \text{if } 1 \le x \le 2 \text{ and } 0 \le y \le x, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Evaluate the constant a.
   (b) Determine the marginal PDF $f_Y(y)$.
   (c) Determine the conditional expectation of $1/X$ given that $Y = 3/2$.
   (d) Random variable Z is defined by $Z = Y - X$. Determine the PDF $f_Z(z)$.

2. Let X and Y be two independent random variables. Their probability densities functions are shown below.



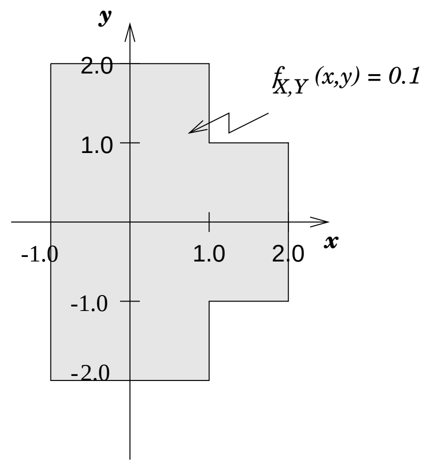


   Let $Z = X + Y$. Determine $f_Z(z)$.

3. Consider n independent tosses of a k-sided fair die. Let X_i be the number of tosses that result in i.

   (a) Are X_1 and X_2 uncorrelated, positively correlated, or negatively correlated? Give a one-line justification.
   (b) Compute the covariance $\text{cov}(X_1, X_2)$ of X_1 and X_2.

4. Random variables X and Y have the joint PDF shown below:



   (a) Find the conditional PDFs $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$, for various values of x and y, respectively.
   (b) Find $\mathbf{E}[X \mid Y = y]$, $\mathbf{E}[X]$, and $\text{var}(X \mid Y = y)$. Use these to calculate $\text{var}(X)$.
   (c) Find $\mathbf{E}[Y \mid X = x]$, $\mathbf{E}[Y]$, and $\text{var}(Y \mid X = x)$. Use these to calculate $\text{var}(Y)$.

5. The wombat club has N members, where N is a random variable with PMF

$$p_N(n) = p^{n-1}(1-p) \quad \text{for } n = 1, 2, 3, \ldots.$$

On the second Tuesday night of every month, the club holds a meeting. Each wombat member attends the meeting with probability q, independently of all the other members. If a wombat attends the meeting, then it brings an amount of money, M, which is a continuous random variable with PDF

$$f_M(m) = \lambda e^{-\lambda m} \quad \text{for } m \geq 0.$$

N, M, and whether each wombat member attends are all independent. Determine:

   (a) The expectation and variance of the number of wombats showing up to the meeting.
   (b) The expectation and variance for the total amount of money brought to the meeting.

G1†. (a) Let $X_1, X_2, \ldots, X_n, X_{n+1}, \ldots, X_{2n}$ be independent and identically distributed random variables.
Find

$$\mathbf{E}[X_1 \mid X_1 + X_2 + \ldots + X_n = x_0],$$

where x_0 is a constant.

   (b) Define

$$S_k = X_1 + X_2 + \ldots + X_k, 1 \leq k \leq 2n.$$

Find

$$\mathbf{E}[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots, S_{2n} = s_{2n}],$$

where $s_n, s_{n+1}, \ldots, s_{2n}$ are constants.

†Required for 6.431; optional for 6.041