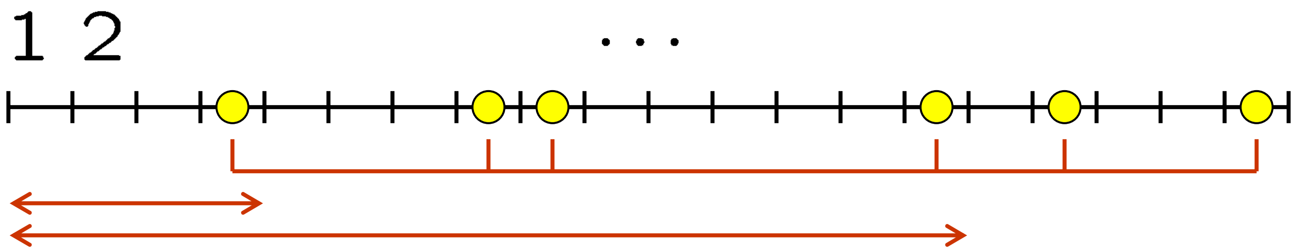
LECTURE 17

- Readings: Start Section 5.2

Lecture outline

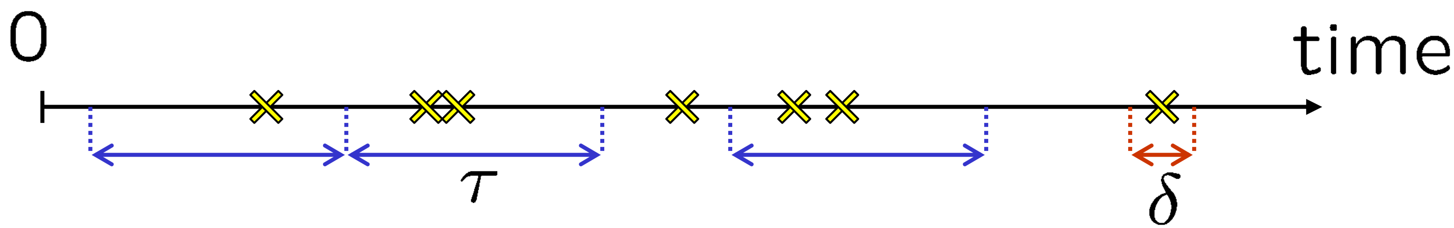
- Review of the Bernoulli process
- Definition of the Poisson process
- Basic properties of the Poisson process
  - Distribution of the number of arrivals
  - Distribution of the interarrival time
  - Distribution of the k^{th} arrival time

The Bernoulli Process: Review



- Discrete time; success probability in each slot $= p$.
- PMF of number of arrivals in n time slots: Binomial
- PMF of interarrival time: Geometric
- PMF of time to k^{th} arrival: Pascal
- Memorylessness
- **What about continuous arrival times?**
  Example: arrival to a bank.

The Poisson Process: Definition

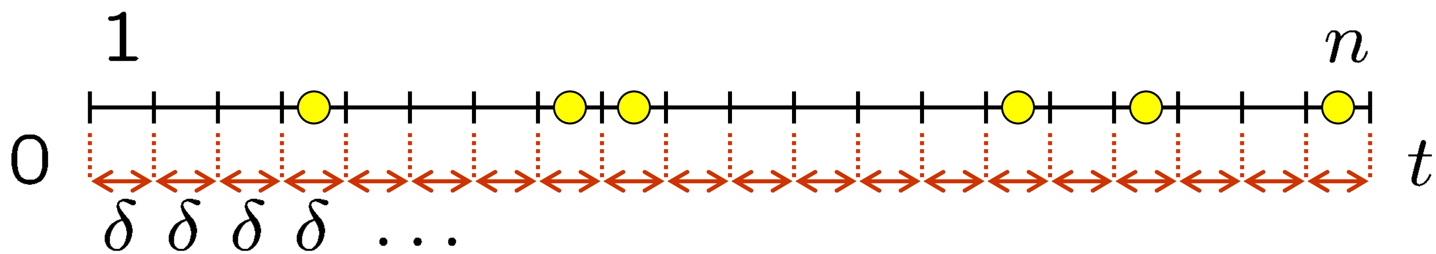


- Let $\mathbf{P}(k, \tau)$ = Probability of k arrivals in an interval of duration τ.
- **Assumptions**:
  - Number of arrivals in disjoint time intervals are independent.
  - For VERY small δ, we have:

$$\mathbf{P}(k, \delta) \approx \begin{cases} 1 - \lambda\delta & \text{if } k = 0 \\ \lambda\delta & \text{if } k = 1 \\ 0 & \text{if } k > 0 \end{cases}$$

  - λ = "arrival rate" of the process.

From Bernoulli to Poisson (1)

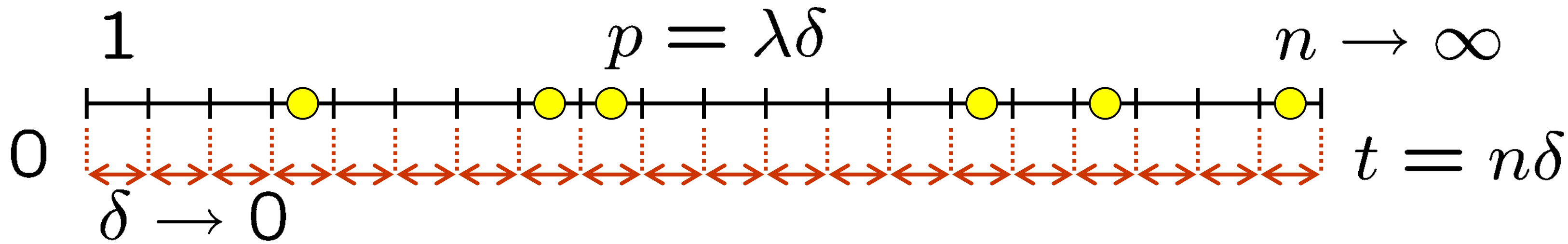


- **Bernoulli:** Arrival prob. in each time slot $= p$
- **Poisson:** Arrival probability in each δ -interval $= \lambda\delta$
- Let $n = t/\delta$ and $p = \lambda\delta$:

Number of arrivals in a t -interval	=	Number of successes in n time slots

(Binomial)

From Bernoulli to Poisson (2)



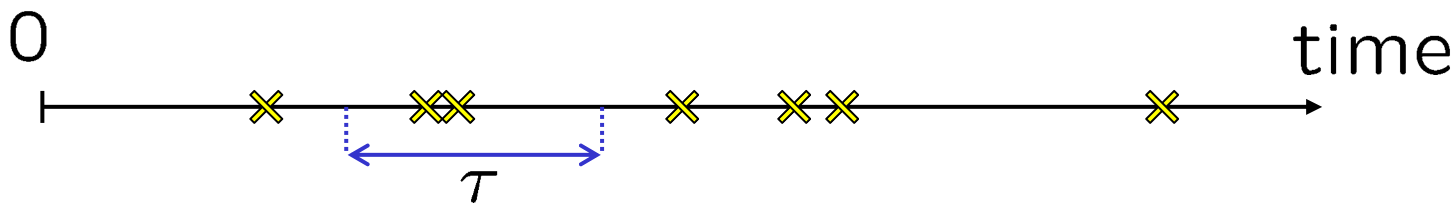
- Number of arrivals in a t-interval as $n \to \infty$ =

$$\binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k} \qquad \text{(Binomial)}$$

$$= \underbrace{\frac{n!}{(n-k)!\,n^k}}_{\downarrow\; 1} \frac{(\lambda t)^k}{k!} \underbrace{\left(1 - \frac{\lambda t}{n}\right)^n}_{\downarrow\; e^{-\lambda t}} \underbrace{\left(1 - \frac{\lambda t}{n}\right)^{-k}}_{\downarrow\; 1} \qquad \text{(reorder terms)}$$

$$= \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad \textbf{(Poisson)}$$

PMF of Number of Arrivals



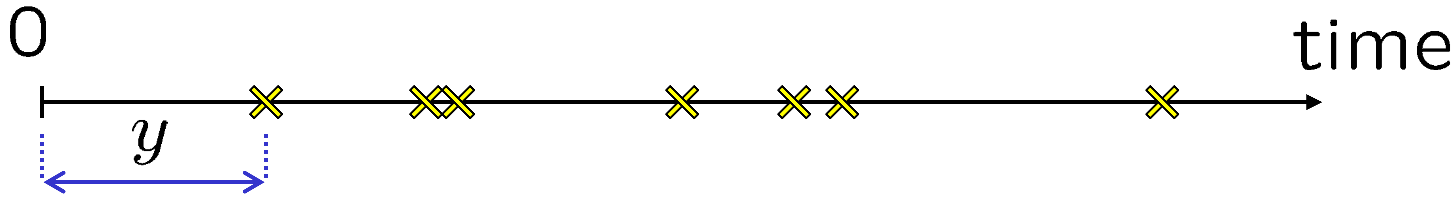
- N : number of arrivals in a τ -interval, thus:
- $\mathrm{P}(N=k)=\mathrm{P}(k,\tau)=\dfrac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$ **(Poisson)**

  $k=0,1,\cdots$
- **Mean**: $\mathrm{E}[N]=\lambda\tau$
- **Variance**: $\mathrm{Var}(N)=\lambda\tau$
- **Transform**: $M_N(s)=e^{\lambda\tau(e^s-1)}$

Email Example

- You get email according to a Poisson process, at a rate of λ = 0.4 messages per hour. You check your email every thirty minutes.
  - Prob. of no new messages = $\frac{(.2)^0 e^{-.2}}{0!} = e^{-.2}$
  - Prob. of one new message = $\frac{(.2)^1 e^{-.2}}{1!} = .2e^{-.2}$

Interarrival Time



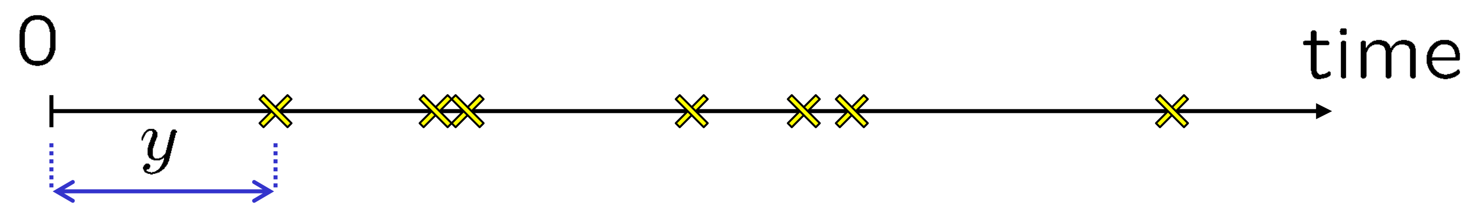
- Y_1 : time of the 1^{st} arrival.
- "First order" interarrival time:

$$f_{Y_1}(y) = \lambda e^{-\lambda y}, \quad y \geq 0$$ **(Exponential)**

- Why:

$$\mathbf{P}(Y_1 \leq y) = 1 - \mathbf{P}(0, y) = 1 - e^{-\lambda y}$$

Interarrival Time

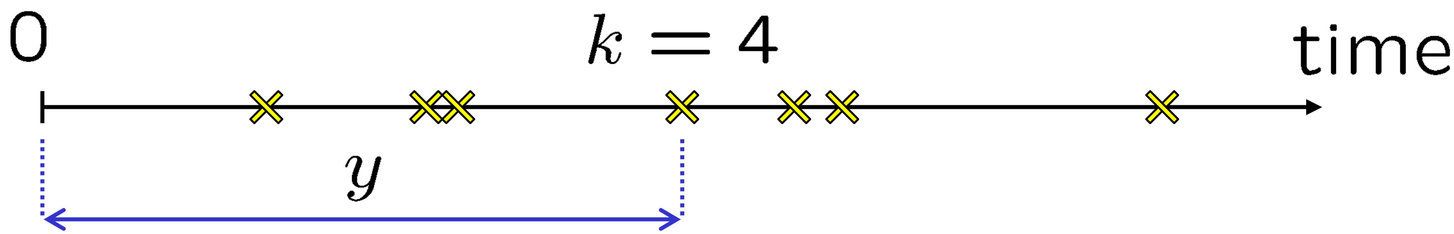


- **Fresh Start Property**: The time of the next arrival is independent from the past.

- **Memoryless property**: Suppose we observe the process for T seconds and no success occurred. Then the density of the remaining time for arrival is exponential.

- **Email Example**: You start checking your email. How long will you wait, in average, until you receive your next email? $\mathbf{E}[Y_1] = \frac{1}{\lambda} = 2.5$ hours

Time of k^{th} Arrival



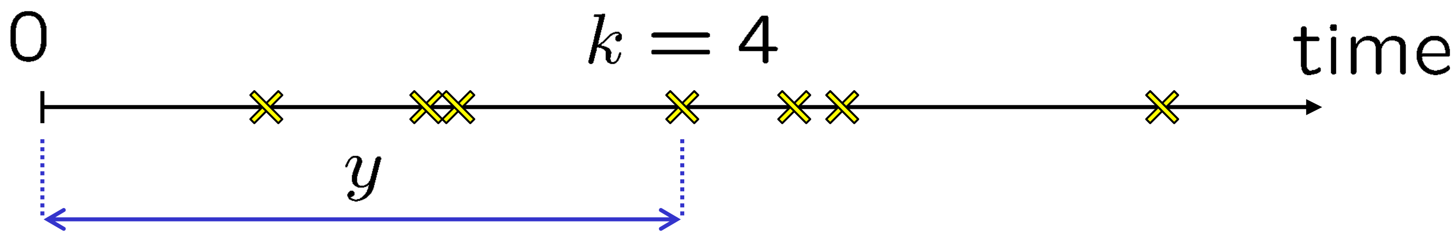
- Y_k : time of the k^{th} arrival.

- $T_k = Y_k - Y_{k-1}$ $k = 2, 3, \ldots$: kth interarrival time

- It follows that:

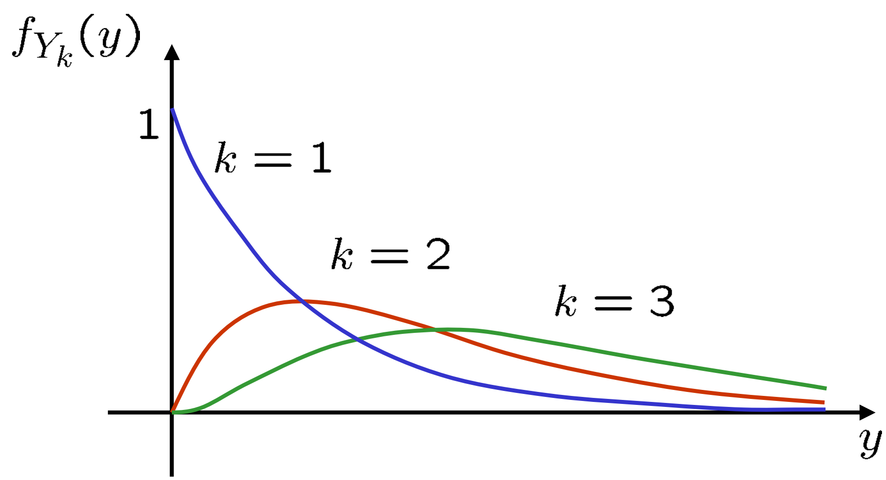
$$Y_k = T_1 + T_2 + \ldots T_k$$

Time of k^{th} Arrival

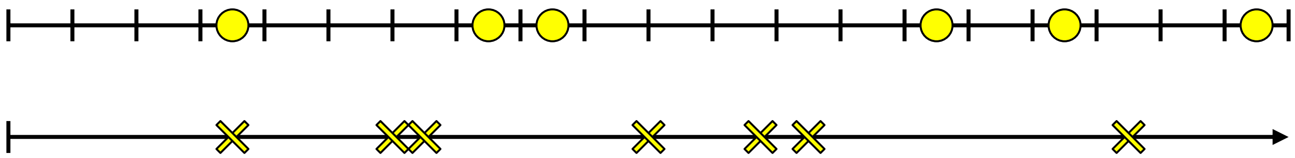


- Y_k : time of the k^{th} arrival.
- $f_{Y_k}(y) = \dfrac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \qquad y \geq 0$

(Erlang)
"of order k"



Bernoulli vs. Poisson



	Bernoulli	Poisson
Times of Arrival	Discrete	Continuous
Arrival Rate	p /per trial	λ /unit time
PMF of Number of Arrivals	Binomial	Poisson
PMF of Interarrival Time	Geometric	Exponential
PMF of k^{th} Arrival Time	Pascal	Erlang